Model-Theoretic Constructions in Many-Valued Modal Logic

by

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Submitted to the
Graduate Program in Logic, Algorithms and Computation (M.P.L.A.)
in partial fulfillment of the requirements for the degree
Master of Science
Department of Mathematics
University of Athens

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Athens
July 2002

Preface

The present thesis constitutes a first investigation on some classical model-theoretic constructions in the context of Many-Valued Modal Logics and it mainly follows the writing format style of a research paper. The subject was suggested by my thesis advisor Dr. Costas Koutras, whose encouragement for pursuing an original work in this field deserves a lot more than my great acknowledgements. Section 4.4 is mainly conducted by myself but what counted as most inspiring for it was our fertile, significant and pleasant cooperation throughout the rest of the paper.

An updated version of this thesis has been submitted for journal publication. The revised version differs from the present account among others in that it contains less extended proofs of many of the theorems included. Moreover, the present thesis insists on some explanatory comments throughout the development of the text. I hope that these conventions will create a more familiar context for all non-specialized in the subject students and other interested people, attracting them in an enjoyable wandering through the next pages. I feel grateful to all Greek people who understand me for obliging them reading an English text. And finally, I am deeply thankful to all students and faculty of M.P.L.A. for making my years at Athens such a great experience and for providing me with a significant guidance and orientation for my academic aspirations.

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Abstract

In this paper we examine the 'big four' frame constructions of modal model theory, in the context of the family of many-valued modal logics introduced by M. Fitting in 1992. Every modal language of this family is built on an underlying space of truth values, a Heyting algebra $\mathcal{H}$. The many-valued generalizations of disjoint unions, generated subframes and bounded morphisms constitute fairly general $\mathcal{H}$-indexed families of frame constructions. Technically, the most interesting construction is the 'algebraic' generalization of the notion of the canonical (or ultrafilter) extension of a frame which makes use of an involved construction inspired from Fitting's canonical model arguments in [7]. We prove general truth invariance and frame validity (preservation) results and apply them to obtain non-definability results for logical modalities and classes of frames. For finite Heyting algebras, many-valued languages possess an equivalent semantics applying to epistemic situations with many interrelated experts and we briefly discuss the interpretation of our results with respect to this semantics.

Contents

1 Introduction

Contemporary Modal Logic has been seriously influenced by the applications it has found in Computer Science, Artificial Intelligence, Formal Linguistics and Philosophy [2, 13]. The key to the success of Modal Logic in so many disciplines is the underlying relational semantics, whose importance for the development of the field can hardly be overstated. The modern trend is to view modal languages as simple languages for talking about relational structures, a formal device that crops over everywhere in Theoretical Computer Science, Knowledge Representation and Cognitive Science. The reader can obtain a picture of the field in the recent books [2, 19, 4] and the survey article [11].

One of the novel approaches in Modal Logic, rather overlooked hitherto, is the family of many-valued modal logics (mv-MLs) introduced by M. Fitting in 1992 [6]. In this approach, Modal Logic is erected on a version of many-valued propositional logic (instead of classical two-valued propositional logic), but in a particular way. Each logic of this family is built on an underlying Heyting algebra $\mathcal{H}$ which provides the space of truth values. Moreover, the elements of $\mathcal{H}$ are encoded directly in the modal language as special constants and thus we can manipulate them explicitly. Perhaps the most novel feature of this family of logics is the version of relational semantics it possesses: the possible-worlds frames interpreting a language of this family are directed graphs whose edges are labelled with an element of the underlying Heyting algebra $\mathcal{H}$ (henceforth called $\mathcal{H}$-frames). This structure can be conceived as a possible-worlds frame with a many-valued accessibility relation or as a special kind of labelled transition system. Semantically, any complete Heyting algebra can serve as the space of truth values but in order to provide a completeness theorem with respect to a Gentzen-style sequent calculus, a finiteness assumption has been employed by M. Fitting in [7]. Note that this family of logics possesses also a tableau proof procedure [9], modal non-monotonic counterparts [8, 18, 15, 17] and has been recently investigated from the Completeness Theory [16] and the Correspondence Theory perspective [14].

This paper deals with the model theory of the Fitting family of many-valued modal logics, focusing on the 'big four' frame constructions and their associated model ones: *disjoint unions, generated subframes* (and *submodels*), *bounded morphisms* and *canonical extensions*. Historically, the study of frames has dominated the 'classical era' of Modal Logic (see [2, Ch. 1.7]), during which mainly generated submodels and bounded morphisms were used as tools for transforming the canonical model in order to prove frame completeness results; the latter construction appears usually in the literature as p-morphism, a truncation of pseudo-epimorphism. The subsequent explosion of modal expressivity investigations brought into foreground the constructions of disjoint unions and canonical extensions (also called ultrafilter extensions), which are also useful in another pillar of modal wisdom, the duality theory (see [11, 2, 19] for results and references). The justification for naming them the 'big four' constructions in the frame-based analysis of modal logic stems partly from the fact that these constructions occur in the celebrated Goldblatt-Thomason theorem which characterizes the elementarily definable classes of frames that are modally definable. Here, we lift these constructions to the many-valued modal languages. For the first three case we can elaborate on $\mathcal{H}$-indexed families of constructions which assert invariance results of a special form: for a fixed truth value $a \in \mathcal{H}$, what remains invariant is the (lattice theoretic) meet of the truth value with a. Of course, the classical notion of truth invariance is a special case for $a = \top$ (the top element of $\mathcal{H}$). We also obtain frame validity results and non-definability applications of these theorems; note that these results are valid for every *complete* Heyting algebra $\mathcal{H}$. The canonical extension case seems to be the most interesting one, at least mathematically. The construction requires the reworking of many relevant notions (full complex algebra, ultrafilter, modal operators in an algebra etc.) from the classical case (which involves the simple boolean lattice 2) to the general case of an arbitrary *finite* Heyting algebra and proceeds to an involved argument inspired from Fitting's canonical model construction in [7]. The main theorem is an algebraic generalization of the classical truth invariance theorem in principal ultrafilters of the powerset algebra and a validity corollary. En route, we discuss also the related areas of algebras and general frames for these languages.

For finite Heyting algebras, the family of logics we investigate here has an equivalent semantics of interest to Knowledge Representation. Actually, these logics can be seen as the formal device for calculating the 'epistemic consensus' of a finite group of experts, interrelated through a 'dominance' relation. In this context, our results have an interesting interpretation that provides additional motivation: we actually define transformations for multiple expert situations that guarantee to preserve (or keep invariant) the epistemic consensus of a specific subset of the experts on the stage. We briefly explain this aspect in a separate section.

The paper is organized as follows: In Section 2 we provide the mathematical background and in Section 3 the syntax and semantics of many-valued modal languages. Section 4 contains the results of this paper, treating the four frame-theoretic constructions. In Section 5 we briefly expose the multiple-expert semantics for many-valued modal languages and the paper ends in Section 6 with a discussion of our results and possible future extensions.

2 Mathematical Preliminaries

In this section we briefly expose the mathematical background for developing the model theory of many-valued modal logics. Although we provide almost all definitions and properties, some familiarity of the reader with the elements of lattice theory (and universal algebra) is assumed; material in this (and in the next) section should better be seen as fixing the notation and the terminology. For further study on these, the reader should refer to [5, 3], whereas Heyting algebras (hereinafter HAs) and their properties are thoroughly presented in [20] and/or [1].

A Heyting algebra is a lattice with some additional properties. A lattice can be defined either algebraically or employing as primitive the notion of *order*. We follow the second approach, as we find it more appealing in the kind of logical applications we pursue. A *lattice* $\mathcal{L}$ is a pair $\langle L, \leq \rangle$, consisting of a non-empty set L equipped with a partial-order relation $\leq$ (i.e. reflexive, transitive and antisymmetric), such that every two-element subset $\{a, b\}$ of L has a *least upper bound* or *join*, denoted by $a + b$, and a *greatest lower bound* or *meet*, denoted by $a \cdot b$. A *filter* in a lattice is a non-empty, upwards-closed subset of the lattice, which is closed under meet. An element $x \in L$ is *join-irreducible* iff $x \neq \bot$ (in case L has a bottom element) and $x = a + b$, implies $x = a$ or $x = b$. With a harmless abuse of notation we do not distinguish between $\mathcal{L}$ and L. A *least* element of a lattice is denoted by $\bot$ and a *greatest* one by $\top$. In any lattice with a least and a greatest element, an element x satisfying $x \cdot a = \bot$ and $x + a = \top$ is called a *complement* of a. A lattice $\mathcal{L}$ is *complete* if a join and a meet exist for *every* subset of $\mathcal{L}$. We frequently use indexed sets and denote (possibly infinite) meets and joins by $\prod_{t \in T} a_t$ and $\sum_{t \in T} a_t$. Some fairly obvious properties of infinite joins and meets will be used generally without mentioning:

Lemma 2.1 *If S and T are indexed sets and $\prod\limits_{t \in T} a_t$, $\sum\limits_{t \in T} a_t$ $\prod\limits_{t \in S} b_t$, $\sum\limits_{t \in S} b_t$ exist in a lattice $\mathcal{L}$, then for all $a, b \in \mathcal{L}$, $\prod\limits_{t \in T} \Big(a + a_t\Big)$, $\sum\limits_{t \in T} \Big(a + a_t\Big)$ also exist in $\mathcal{L}$ and the following properties hold as well:*

1. $\prod\limits_{t \in T} \Big(a \cdot a_t\Big) \; = \; a \cdot \prod\limits_{t \in T} a_t$
2. $\sum\limits_{t \in T} \Big(a + a_t\Big) \; = \; a + \sum\limits_{t \in T} a_t$
3. $\prod\limits_{t \in T} a_t \cdot \prod\limits_{t \in T} b_t \; = \; \prod\limits_{t \in T} a_t \cdot b_t$
4. $b \leq \prod\limits_{t \in T} a_t$ *iff* $b \leq a_t$ *for every* $t \in T$
5. $\prod\limits_{t \in T} a_t \leq a_{t_0}$ *for every* $t_0 \in T$
6. *If* $a_t \leq b_t$ *for every* $t \in T$, *then* $\prod\limits_{t \in T} a_t \; \leq \; \prod\limits_{t \in T} b_t$ *and* $\sum\limits_{t \in T} a_t \; \leq \; \sum\limits_{t \in T} b_t$
7. *If* $S \subseteq T$, *then* $\prod\limits_{t \in T} b_t \; \leq \; \prod\limits_{s \in S} a_s$ *and* $\sum\limits_{s \in S} a_s \; \leq \; \sum\limits_{s \in T} b_t$

A lattice $\mathcal{H} = \langle H, \leq \rangle$ with the additional property that, for every pair of elements $\langle a, b \rangle$, the set $\{x \mid a \cdot x \leq b\}$ has a greatest element, is called a *relatively pseudo-complemented lattice* (short: rpc-lattice). This element is denoted by $a \Rightarrow b$ and is called the *pseudo-complement of a relative to b*. The definition of relative pseudo-complement is equivalent to the existence of an element $a \Rightarrow b$ satisfying the fundamental property

$$\mathcal{H}_1 : \; x \leq \Big(a \Rightarrow b\Big) \quad \text{iff} \quad \Big(x \cdot a\Big) \leq b$$

Since it is always the case that $a \cdot x \leq a$, it follows by property $\mathcal{H}_1$ that $x \leq (a \Rightarrow a)$ for any $x, a \in \mathcal{H}$; thus, an rpc-lattice is a topped ordered set: just set $\top = (a \Rightarrow a)$ for an arbitrary $a \in H$. However, it is not always the case that an rpc-lattice has a least element.

Definition 2.2 An rpc-lattice $\mathcal{H}$ with a least element is called a *Heyting algebra* (HA) or *pseudo-Boolean algebra.*

In a HA, we define the *negation* of a as $a' = (a \Rightarrow \bot)$. It is directly derived that $a \cdot a' = \bot$ but, in general it does not hold that $a + a' = \top$; this explains why a' is also called the *pseudo-complement* of a. This choice for the definition of a' can be further justified by the following fact: if an element c has a complement, that complement must be c'.

The following properties of rpc-lattices (and thus, of HAs) will be used in our proofs:

$$\mathcal{H}_2 : \ \Big(a \Rightarrow b\Big) = \top \quad \text{iff} \quad a \leq b$$

$$\mathcal{H}_3 : \ \Big(\top \Rightarrow a\Big) \ = \ a$$

$$\mathcal{H}_4 : \ \text{If} \ \ a_1 \leq a_2 \quad \text{then} \quad \Big(a_2 \Rightarrow b\Big) \leq \Big(a_1 \Rightarrow b\Big)$$

$$\mathcal{H}_5 : \ \text{If} \ \ b_1 \leq b_2 \quad \text{then} \quad \Big(a \Rightarrow b_1\Big) \leq \Big(a \Rightarrow b_2\Big)$$

$$\mathcal{H}_6 : \ \Big(a \Rightarrow b\Big) \cdot \Big(a \Rightarrow c\Big) \ = \ \Big(a \Rightarrow (b \cdot c)\Big)$$

$$\mathcal{H}_7 : \ \Big(a \Rightarrow c\Big) \cdot \Big(b \Rightarrow c\Big) \ = \ \Big((a + b) \Rightarrow c\Big)$$

$$\mathcal{H}_8 : \ \Big(a \Rightarrow (b \Rightarrow c)\Big) \ = \ \Big((a \cdot b) \Rightarrow c\Big) \ = \ \Big(b \Rightarrow (a \Rightarrow c)\Big)$$

$$\mathcal{H}_9 : \ \Big(a \Rightarrow b\Big) \cdot \Big(b \Rightarrow c\Big) \ \leq \ \Big(a \Rightarrow c\Big)$$

$$\mathcal{H}_{10} : \ \ c \cdot \Big(a \Rightarrow b\Big) \ = \ c \cdot \Big((c \cdot a) \Rightarrow (c \cdot a)\Big)$$

Three more properties are given in the next lemma, where $\big(a \Leftrightarrow \top\big)$ is an abbreviation of $\big(a \Rightarrow \top\big) \cdot \big(\top \Rightarrow a\big)$.

Lemma 2.3 *1.* $\Big(a \Leftrightarrow \top\Big) \ = \ a$

2. $a \ \leq \ \Big(\big(a \Rightarrow b\big) \Rightarrow b\Big)$

PROOF. 1. By $\mathcal{H}_2$, $\big(a \Rightarrow \top\big) = \top$. $\mathcal{H}_3$ says $\big(\top \Rightarrow a\big) = a$. Hence, $\big(a \Leftrightarrow \top\big) = \big(a \Rightarrow \top\big) \cdot \big(\top \Rightarrow a\big) = \top \cdot a = a$.
2. Directly from the definition of $\Rightarrow$. ∎

In a Boolean algebra, the operation of relative pseudo-complementation can be retrieved as $(a \Rightarrow b) = a' + b$. It can be also proved that HAs are distributive. Thus, the class of HAs includes the class of Boolean algebras and is included in the class of distributive lattices; both inclusions are proper [20]. If however we restrict attention to finite lattices, the second inclusion becomes an equality: it can be shown that in each finite distributive lattice the operation $(a \Rightarrow b)$ can be defined and there always exists a least element $\perp$. Thus, the class of finite HAs coincides with the class of finite distributive lattices.

The following useful properties of rpc-lattices (and thus, of HAs) employ (possibly) infinite joins and meets.

Lemma 2.4 *If T is an indexed set and $\prod_{t \in T} a_t$, $\sum_{t \in T} a_t$, $\prod_{t \in T} b_t$, $\sum_{t \in T} b_t$ exist in an rpc-lattice $\mathcal{H}$, then all the infinite meets and joins below also exist in $\mathcal{H}$ and the properties hold as well.*

$$\mathcal{H}_1^{\cdot} : \quad \prod_{t \in T} (a \Rightarrow b_t) = \Big(a \Rightarrow \prod_{t \in T} b_t\Big)$$

$$\mathcal{H}_2^{\cdot} : \quad \prod_{t \in T} (a_t \Rightarrow b) = \Big(\big(\sum_{t \in T} a_t\big) \Rightarrow b\Big)$$

$$\mathcal{H}_1^{+} : \quad \sum_{t \in T} (a \cdot b_t) = \Big(a \cdot \sum_{t \in T} b_t\Big)$$

Remark 2.5 Proofs of the properties of rpc-lattices and HAs mentioned in this section can be found in [20, Ch. I & IV] and [1, Ch. IX].
Let us also argue about the non-standard notation we follow here regarding the boolean meets and joints, as one would expect to see the symbols $\wedge, \vee, \bigwedge, \bigvee$ instead of $\cdot, +, \prod, \sum$. The reason for doing so is the variety of roles that the former classical symbols play in this paper; we use $\wedge$ and $\vee$ as the language connectives but we avoid using them to denote operations of HAs (we introduce another HA in Def. 4.13).

We close this section with some facts from the universal algebra perspective. Viewed algebraically, a HA $\mathcal{H}$ is an algebra of signature $\langle H, +, \cdot, \Rightarrow, 0\rangle$ which satisfies certain equations [1, Ch. IX.4]. This means of course that the class of HAs is a variety. Given an indexed set I, the power $\mathcal{H}^I$ is the Heyting algebra with carrier set H^I and whose operations are coordinate-wise defined. We will use this structure in Section 4.4 to define the canonical extension of many-valued modal frames.

3 Syntax and Semantics of Many-Valued Modal Logics (mv-MLs)

In order to define a language of propositional mv-ML, we first fix a complete HA $\mathcal{H}$ which will serve as the space of truth values (this justifies us speaking of a family of mv-MLs). The elements of $\mathcal{H}$ are directly represented in the language by special *propositional constants*, and we reserve lowercase letters (along with $\bot, \top$) to denote them. To facilitate notation, we use the same letter for the element of $\mathcal{H}$ and the constant which represents it in the language; context will clarify what is meant. The existence of $\mathcal{H}$-truth values in the language is very fruitful as it allows for making explicit statements for the relative truth value of formulas.

Having fixed a HA $\mathcal{H}$, we choose a set Φ of *propositional variables* to construct the *(basic) many-valued modal language* $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$. It is customary to consider a countable set Φ, but note that the cardinality of the language (which in our case is also affected by the cardinality of $\mathcal{H}$) will not play a role in this paper. [1] The set of *atomic formulas* of $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$ is the union of the set Φ of propositional variables and the set of propositional constants (the $\mathcal{H}$-truth values). The set of *well-formed formulas* of $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$ is generated by the set of atomic formulas using the formation rules for the propositional connectives $\vee, \wedge, \supset, \neg$ and the modal operators $\Box, \Diamond$. In the BNF specification following, a ranges over elements of $\mathcal{H}$, P ranges over elements of Φ and A is a formula of $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$.

$$A \ ::= \ a \ | \ P \ | \ A_1 \vee A_2 \ | \ A_1 \wedge A_2 \ | \ A_1 \supset A_2 \ | \ \neg A \ | \ \Box A \ | \ \Diamond A$$

It is also usual to abbreviate $(A_1 \supset A_2) \wedge (A_2 \supset A_1)$ by $(A_1 \equiv A_2)$. In the rest of the paper, we will omit Φ when possible and speak of the language $L^{\mathcal{H}}_{\Box\Diamond}$. The above definition of $L^{\mathcal{H}}_{\Box\Diamond}$ is superfluous in that the (non-classical) negation $\neg X$ can be defined as $(X \supset \bot)$, but it is harmless to add it explicitly.

Now, we turn to the semantics of many-valued modal languages. The semantics described below alters in a very interesting manner the notion of relational frames, in that it concerns a kind of many-valued accessibility relation. Note that there have been other approaches in the literature for defining many-valued modal logics, but all of them have kept the essence of classical relational semantics intact (see [7] for references). Given a language $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$, we define $\mathcal{H}$-modal frames and $\mathcal{H}$-modal models as follows. We shall henceforth drop the adjective 'modal' and talk simply of $\mathcal{H}$-frames and $\mathcal{H}$-models.

Definition 3.1 An *$\mathcal{H}$-modal frame* for $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$ is a pair $\mathfrak{F} = \langle \mathfrak{S}, \mathfrak{g} \rangle$, where $\mathfrak{S}$ is a non-empty set of *states* and $\mathfrak{g} : \mathfrak{S} \times \mathfrak{S} \to \mathcal{H}$ is a total function mapping pairs of states to elements of $\mathcal{H}$.
An *$\mathcal{H}$-modal model* $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ is built on $\mathfrak{F}$ by providing a *valuation* v, that is a function $v : \mathfrak{S} \times (\mathcal{H} \cup \Phi) \to \mathcal{H}$ which assigns an *$\mathcal{H}$-truth value* to atomic formulas in each state, such that $v(\mathfrak{s}, a) = a$, for every $\mathfrak{s} \in \mathfrak{S}$ and $a \in \mathcal{H}$. That is, propositional constants are always mapped to 'themselves'.

We can now readily extend the valuation v to all the formulas of $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$ (for all states of $\mathfrak{M}$ simultaneously) in a standard recursive fashion. Abusing notation, we denote by v both the initial valuation and its extension to all formulas of $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$.

Definition 3.2 Let $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ be an *$\mathcal{H}$-model*. The extension of the valuation v is given by the following clauses:

[1] The cardinality of the language is of significant interest to important metamathematical problems in modal logic, concerning the relation between *canonicity* of a modal logic and its *completeness with respect to elementarily definable classes of frames* [10, 12].

- $v(\mathfrak{s}, A \wedge B) = \Big(v(\mathfrak{s}, A) \cdot v(\mathfrak{s}, B)\Big)$
- $v(\mathfrak{s}, A \vee B) = \Big(v(\mathfrak{s}, A) + v(\mathfrak{s}, B)\Big)$
- $v(\mathfrak{s}, A \supset B) = \Big(v(\mathfrak{s}, A) \Rightarrow v(\mathfrak{s}, B)\Big)$
- $v(\mathfrak{s}, \neg A) \quad = \quad \Big(v(\mathfrak{s}, A) \Rightarrow \bot\Big)$
- $v(\mathfrak{s}, \Box A) \quad = \quad \prod_{\mathfrak{t} \in \mathfrak{S}} \Big(\mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \Rightarrow v(\mathfrak{t}, A)\Big)$
- $v(\mathfrak{s}, \Diamond A) \quad = \quad \sum_{\mathfrak{t} \in \mathfrak{S}} \Big(\mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \cdot v(\mathfrak{t}, A)\Big)$

We pause for an important remark: in the introductory section, we did speak of an $\mathcal{H}$-frame $\mathfrak{F}$ as a labelled directed graph. However, Definition 3.1 provides a complete labelled directed graph as $\mathfrak{g}$ is a total function attaching an $\mathcal{H}$-truth value to *every* edge. But, the reader can readily check that, by the last clauses of Def. 3.2 above, along with property $\mathcal{H}_2$, the edges labelled by $\bot$ do not play a role in the calculation of the truth value of modal formulas. Thus, an edge labelled $\bot$ does not really mean any kind of 'relative accessibility' and can be as well deleted from the labelled graph of $\mathfrak{F}$.

There is also another remark, whose key role for the rest of the paper urges on isolating it:

Remark 3.3 We have implicitly begun following a principle, which enables us to pass from modal logic to many-valued modal logic. We did it when we involved a function $\mathfrak{g}$ in Def. 3.1 to play the role of the 'accessibility relation' of the classical two-valued case. This *'functions instead of sets' principle* will further be invoked in Section 4.4, where, for instance, we lift 'the set of all states where φ is true' to a function which assigns the value $v(\mathfrak{s}, \varphi) \in \mathcal{H}$ to every state $\mathfrak{s}$. It should be stressed that if we apply the 'functions instead of sets' principle remaining in a two-valued basis, we just obtain the characteristic function corresponding to the set in question.

Now, possessing a way to calculate the $\mathcal{H}$-truth value of every formula in each state of a model, we can proceed to define the principal notions of *truth* and *validity*.

Definition 3.4 A formula A of $L^{\mathcal{H}}_{\Box\Diamond}$ is *true in a state* $\mathfrak{s}$ of an $\mathcal{H}$-model $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ (denoted as $\mathfrak{M}, \mathfrak{s} \Vdash A$) if $v(\mathfrak{s}, A) = \top$; we also say that A is *satisfied* in $\mathfrak{s}$. Further on, A is *true in a model* $\mathfrak{M}$ (denoted as $\mathfrak{M} \Vdash A$) if $v(\mathfrak{t}, A) = \top$ for every $\mathfrak{t} \in \mathfrak{S}$, and it is *valid* in an $\mathcal{H}$-frame $\mathfrak{F}$ (denoted as $\mathfrak{F} \Vdash A$) if it is true in every $\mathcal{H}$-model built on $\mathfrak{F}$. The notions of truth and validity extend in a class C of $\mathcal{H}$-models or $\mathcal{H}$-frames, in the obvious way (notation: $\mathsf{C} \Vdash A$).

It shall be later necessary to speak of a more refined notion of truth (or validity) of a formula in an $\mathcal{H}$-model (or $\mathcal{H}$-frame):

Definition 3.5 A modal formula X is called *a-true* (*a-valid*) in an $\mathcal{H}$-model $\mathfrak{M}$ ($\mathcal{H}$-frame $\mathfrak{F}$) if the formula $(a \supset X)$ is true in $\mathfrak{M}$ (valid in $\mathfrak{F}$). Notation: $\mathfrak{M} \Vdash_a X$ ($\mathfrak{F} \Vdash_a X$).

We will mostly use the refined a-validity of X in an $\mathcal{H}$-frame, when we talk about (a-)preserving validity results for frames. Let us now informally introduce a notion which will concern us in the next section and will become formal en route.

Definition 3.6 A property P (like truth or validity) is called *a-invariant* (or *invariant with-respect-to-a*) under a construction F if $a \cdot P = a \cdot F(P)$, where $F(P)$ is the corresponding property of P after applying the construction F.

Following is the classical definition of correspondence between a modal formula and a class of frames. Equivalently, we speak of modal definability of a property of $\mathcal{H}$-frames.

Definition 3.7 Let C be a class of $\mathcal{H}$-frames and φ a formula of $L^{\mathcal{H}}_{\Box\Diamond}$. We say that φ *defines* C iff for every $\mathcal{H}$-frame $\mathfrak{F}$, $\mathfrak{F} \in \mathsf{C}$ iff $\mathfrak{F} \Vdash \varphi$. If C is the class of $\mathcal{H}$-frames satisfying a property P, we say that φ *corresponds to* or *defines* property P. A class (or a property) of $\mathcal{H}$-frames is called *modally definable* if there exists a $\varphi \in L^{\mathcal{H}}_{\Box\Diamond}$ which defines it.

Finally, we have three more remarks:

Remark 3.8 The significance of coinciding propositional constants with the elements they represent is clear in the definition 3.5, where we were able to speak about the relative truth and validity of formulas. If we were working in the classical two-valued case, we wouldn't have to introduce these propositional constants: the elements $\top$ and $\bot$ are well definable within the language by $X \vee \neg X$ and $X \wedge \neg X$, for an arbitrary formula X. Further on, in mv-MLs we are forced to introduce both $\Box$ and $\Diamond$ as primitive connectives, since it can be shown ([7]) that they are not each other's dual ($\Box A \equiv \neg\Diamond\neg A$ is not valid in all $\mathcal{H}$-models), unless $\mathcal{H}$ is a Boolean algebra. That is, many-valued modal language is *bimodal*, a characteristic feature of mv-MLs.

Remark 3.9 If we choose $\mathcal{H}$ to be $\mathbf{2}$ (the well-known lattice of classical logic , denoted here for homogeneity as $\bot \leq \top$), then we obtain classical two-valued propositional modal logic. It is an easy exercise to verify that all the definitions and results of this section (and of the whole paper for that matter) for the language $L^{\mathbf{2}}_{\Box\Diamond}$ collapse to the familiar notions and results from the classical case. In this sense, the family of mv-MLs we examine constitutes a multi-dimensional generalization of classical logic, both at the propositional as well as at the modal dimension.

Remark 3.10 The notation we have fixed in this section will be used throughout the paper. We reserve $\mathcal{H}$ for the fixed HA on which the language $L^{\mathcal{H}}_{\Box\Diamond}$ is built and we use lowercase letters $a, b, c, \ldots, t$ to denote its elements and the corresponding propositional constants. The only notable exception is v, which is reserved for the valuation in an

$\mathcal{H}$-model. We use uppercase fraktur letters $\mathfrak{F}$, $\mathfrak{M}$ for frames and models for $L^{\mathcal{H}}_{\Box\Diamond}$. The set of states of an $\mathcal{H}$-frame is always denoted by $\mathfrak{S}$ and we use the lowercase fraktur letters $\mathfrak{s}, \mathfrak{r}, \mathfrak{t}$ as variables ranging over its members. The symbol $\mathfrak{g}$ is used throughout the paper to denote the 'accessibility' function in an $\mathcal{H}$-frame. Finally, note that since we only deal with *propositional* mv-ML, we can use symbols $\forall$ and $\exists$ for abbreviating the english expressions 'for every' and 'there exists'.

4 Model and Frame Constructions in mv-ML

The constructions we deal with are the 'big four' ones considered in Chapters 2 & 3 of [2]. In their many-valued versions, we are still interested in separating the model from the frame level. The invariance results derived from the first one will be helpful in pointing out some logical modalities not definable in our language $L^{\mathcal{H}}_{\Box\Diamond}$, whereas in the frame level, certain classes (or properties) of frames will be shown to be not modally definable. In each case, it is easy to check that the definitions and the results are proper generalizations of the corresponding ones in the two-valued case, by simply replacing $\mathcal{H}$ by $\mathbf{2}$ and a by $\top$. In this sense, the canonical extension construction constitutes an exception, since no general form of it involving a is presented, nor non-definability results are derived. This construction will again be an exception in section 5, where an interpretation from Knowledge Representation area will be given among others to the role of a in our results, but not of course for the canonical extension case.

Let us now be more specific and stress some special points of the presentation to follow. In each section we first define an *a-(preserving) construction* on model and frame levels. Then, two theorems concerning *a-invariance results* in both levels are in order. These are truth invariance and frame validity preservation results. Some of them will roughly have the following typical form:

- Let $\mathfrak{M}' = \langle \mathfrak{S}', \mathfrak{g}', v' \rangle$ be the result of an a-construction applied on an $\mathcal{H}$-model $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$. Then, for every formula φ of $L^{\mathcal{H}}_{\Box\Diamond}$ and state $\mathfrak{s}$ of $\mathfrak{M}'$,

$$a \cdot v(\mathfrak{s}, \varphi) = a \cdot v'(\mathfrak{s}, \varphi).$$

- Let $\mathfrak{F}' = \langle \mathfrak{S}', \mathfrak{g}' \rangle$ be the result of an a-construction applied on an $\mathcal{H}$-frame $\mathfrak{F} = \langle \mathfrak{S}, \mathfrak{g} \rangle$. Then, for every formula φ of $L^{\mathcal{H}}_{\Box\Diamond}$,

$$\mathfrak{F} \Vdash_a \varphi \text{ iff } \mathfrak{F}' \Vdash_a \varphi.$$

Note that the second of the above statements turns out (easily) to be equivalent to the following:

- Let $\mathfrak{F}' = \langle \mathfrak{S}', \mathfrak{g}' \rangle$ be the result of an a-construction applied on an $\mathcal{H}$-frame $\mathfrak{F} = \langle \mathfrak{S}, \mathfrak{g} \rangle$. Then, for every formula φ of $L^{\mathcal{H}}_{\Box\Diamond}$,

$$\mathfrak{F} \Vdash a \wedge \varphi \text{ iff } \mathfrak{F}' \Vdash a \wedge \varphi.$$

Finally, some of these results are used to produce several non-definability results by setting $a = \top$. But, we have gone too far with this introductory part, so let us begin with the presentation of our constructions.

4.1 Disjoint Unions

The first construction to be considered is that of disjoint unions. It is the most simple one and, in fact, in generalizing it we only present the refined many-valued version of some known notions. However, this may be a good way to (warm up and) get introduced in the style of later constructions and results.

Definition 4.1 For an index set I, let $\mathfrak{M}_i = \langle \mathfrak{S}_i, \mathfrak{g}_i, v_i \rangle, (i \in I)$ be a collection of $\mathcal{H}$-models for $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$. We call these models *disjoint* if for every $i, j \in I, \mathfrak{S}_i \cap \mathfrak{S}_j = \emptyset$. The *a-(preserving) $\mathcal{H}$-disjoint union* of a collection of disjoint models $\mathfrak{M}_i = \langle \mathfrak{S}_i, \mathfrak{g}_i, v_i \rangle, (i \in I)$ is the $\mathcal{H}$-model $\biguplus_i \mathfrak{M}_i = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$, where $\mathfrak{S}$ is the union of the domains $\mathfrak{S}_i$ and $\mathfrak{g}$, v are the *trivial expansions* of all $\mathfrak{g}_i$, v_i, respectively:

1. for every $\mathfrak{s} \in \mathfrak{S}_i, \mathfrak{t} \in \mathfrak{S}_j$, $a \cdot \mathfrak{g}(\mathfrak{s}, \mathfrak{t}) = a \cdot \mathfrak{g}_i(\mathfrak{s}, \mathfrak{t})$ if $i = j$, else $a \cdot \mathfrak{g}(\mathfrak{s}, \mathfrak{t}) = \perp$
2. for every $\mathfrak{s} \in \mathfrak{S}_i$ and $p \in \Phi$, $a \cdot v(\mathfrak{s}, p) = a \cdot v_i(\mathfrak{s}, p)$

Similarly, a collection $\mathfrak{F}_i = \langle \mathfrak{S}_i, \mathfrak{g}_i \rangle, (i \in I)$ of $\mathcal{H}$-frames is called *disjoint* if for every $i, j \in I, \mathfrak{S}_i \cap \mathfrak{S}_j = \emptyset$. For a collection $\mathfrak{F}_i = \langle \mathfrak{S}_i, \mathfrak{g}_i \rangle, (i \in I)$ of $\mathcal{H}$-frames, their *a-disjoint union* is the $\mathcal{H}$-frame $\biguplus_i \mathfrak{F}_i = \langle \mathfrak{S}, \mathfrak{g} \rangle$, where $\mathfrak{S} = \bigcup\limits_{i \in I} \mathfrak{S}_i$ and $\mathfrak{g}$ satisfies the above condition 1.

The truth invariance result is stated in the next theorem:

Theorem 4.2 *Let $\biguplus_i \mathfrak{M}_i = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ be the disjoint union of a collection of disjoint $\mathcal{H}$-models $\mathfrak{M}_i = \langle \mathfrak{S}_i, \mathfrak{g}_i, v_i \rangle, (i \in I)$. Then, for each formula A of $L^{\mathcal{H}}_{\Box\Diamond}$ and state $\mathfrak{s}$ of $\mathfrak{M}_i$,*

$$a \cdot v(\mathfrak{s}, A) = a \cdot v_i(\mathfrak{s}, A).$$

That is, modal satisfaction is a-invariant under a-disjoint unions.

Proof. By induction on A (for all states of $\mathfrak{M}_i$ simultaneously):
For $A = p \in \Phi$, by item 2 of Definition 4.1.
For $A = a \in \mathcal{H}$, trivially, since by Def. 3.1, $v(\mathfrak{s}, a) = a = v_i(\mathfrak{s}, a)$.
For $A = A_1 \wedge A_2$,

$$\begin{aligned}
a \cdot v(\mathfrak{s}, A_1 \wedge A_2) &= a \cdot v(\mathfrak{s}, A_1) \cdot v(\mathfrak{s}, A_2) \\
&= \Big(a \cdot v(\mathfrak{s}, A_1)\Big) \cdot \Big(a \cdot v(\mathfrak{s}, A_2)\Big) \\
&= \Big(a \cdot v_i(\mathfrak{s}, A_1)\Big) \cdot \Big(a \cdot v_i(\mathfrak{s}, A_2)\Big) \qquad \text{(by Ind. Hyp.)} \\
&= a \cdot v_i(\mathfrak{s}, A_1) \cdot v_i(\mathfrak{s}, A_2) \\
&= a \cdot v_i(\mathfrak{s}, A_1 \wedge A_2)
\end{aligned}$$

For $A = A_1 \vee A_2$,

$$\begin{aligned}
a \cdot v(\mathfrak{s}, A_1 \vee A_2) &= a \cdot \Big(v(\mathfrak{s}, A_1) + v(\mathfrak{s}, A_2)\Big) \\
&= \Big(a \cdot v(\mathfrak{s}, A_1)\Big) + \Big(a \cdot v(\mathfrak{s}, A_2)\Big) \\
&= \Big(a \cdot v_i(\mathfrak{s}, A_1)\Big) + \Big(a \cdot v_i(\mathfrak{s}, A_2)\Big) && \text{(by Ind. Hyp.)} \\
&= a \cdot \Big(v_i(\mathfrak{s}, A_1) + v_i(\mathfrak{s}, A_2)\Big) \\
&= a \cdot v_i(\mathfrak{s}, A_1 \vee A_2)
\end{aligned}$$

For $A = A_1 \supset A_2$,

$$\begin{aligned}
a \cdot v(\mathfrak{s}, A_1 \supset A_2) &= a \cdot \Big(v(\mathfrak{s}, A_1) \Rightarrow v(\mathfrak{s}, A_2)\Big) \\
&= a \cdot \bigg(\Big(a \cdot v(\mathfrak{s}, A_1)\Big) \Rightarrow \Big(a \cdot v(\mathfrak{s}, A_2)\Big)\bigg) && \text{(by } \mathcal{H}_{10}\text{)} \\
&= a \cdot \bigg(\Big(a \cdot v_i(\mathfrak{s}, A_1)\Big) \Rightarrow \Big(a \cdot v_i(\mathfrak{s}, A_2)\Big)\bigg) && \text{(by Ind. Hyp.)} \\
&= a \cdot \Big(v_i(\mathfrak{s}, A_1) \Rightarrow v_i(\mathfrak{s}, A_2)\Big) && (\mathcal{H}_{10}) \\
&= a \cdot v_i(\mathfrak{s}, A_1 \supset A_2)
\end{aligned}$$

In the not such trivial case of $A = \Box A_1$, we have:

$$\begin{aligned}
a \cdot v(\mathfrak{s}, \Box A_1) &= a \cdot \prod_{\mathfrak{t} \in \mathfrak{S}} \Big(\mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \Rightarrow v(\mathfrak{t}, A_1)\Big) \\
&= \prod_{\mathfrak{t} \in \mathfrak{S}} \bigg(a \cdot \Big(\mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \Rightarrow v(\mathfrak{t}, A_1)\Big)\bigg) \\
&= \prod_{\mathfrak{t} \in \mathfrak{S}} a \cdot \Big(a \cdot \mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \Rightarrow a \cdot v(\mathfrak{t}, A_1)\Big) && \text{(by } \mathcal{H}_{10}\text{)} \\
&= \prod_{\mathfrak{t} \in \mathfrak{S}_i} a \cdot \Big(\underbrace{a \cdot \mathfrak{g}_i(\mathfrak{s}, \mathfrak{t})}_{\text{by Def. 4.1}} \Rightarrow a \cdot v_i(\mathfrak{t}, A_1)\Big) && \text{(by Ind. Hyp.)} \\
&= a \cdot \prod_{\mathfrak{t} \in \mathfrak{S}_i} \Big(\mathfrak{g}_i(\mathfrak{s}, \mathfrak{t}) \Rightarrow v_i(\mathfrak{t}, A_1)\Big) \\
&= a \cdot v_i(\mathfrak{s}, \Box A_1)
\end{aligned}$$

Finally, for $A = \Diamond A_1$, we take in a similar way:

$$\begin{aligned}
a \cdot v(\mathfrak{s}, \Diamond A_1) &= a \cdot \sum_{\mathfrak{t} \in \mathfrak{S}} \Big(\mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \cdot v(\mathfrak{t}, A_1) \Big) \\
&= \sum_{\mathfrak{t} \in \mathfrak{S}} \Big(a \cdot \mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \cdot v(\mathfrak{t}, A_1) \Big) && \text{(by } \mathcal{H}_1^+ \text{)} \\
&= \sum_{\mathfrak{t} \in \mathfrak{S}} \Big(\Big(a \cdot \mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \Big) \cdot \Big(a \cdot v(\mathfrak{t}, A_1) \Big) \Big) \\
&= \sum_{\mathfrak{t} \in \mathfrak{S}_i} \Big(\Big(\underbrace{a \cdot \mathfrak{g}_i(\mathfrak{s}, \mathfrak{t})}_{\text{by Def. 4.1}} \Big) \cdot \Big(a \cdot v_i(\mathfrak{t}, A_1) \Big) \Big) && \text{(by Ind. Hyp.)} \\
&= a \cdot \sum_{\mathfrak{t} \in \mathfrak{S}_i} \Big(\mathfrak{g}_i(\mathfrak{s}, \mathfrak{t}) \cdot v_i(\mathfrak{t}, A_1) \Big) \\
&= a \cdot v_i \Big(\mathfrak{s}, \Diamond A_1 \Big)
\end{aligned}$$

∎

We postpone any non-definability results for the next section. Here, we proceed to the invariance result in the frame level:

Theorem 4.3 *Let $\biguplus_i \mathfrak{F}_i = \langle \mathfrak{S}, \mathfrak{g} \rangle$ be the disjoint union of a collection of disjoint $\mathcal{H}$-frames $\mathfrak{F}_i = \langle \mathfrak{S}_i, \mathfrak{g}_i \rangle, (i \in I)$. Then, for each formula X of $L^{\mathcal{H}}_{\Box\Diamond}$,*

$$\biguplus_i \mathfrak{F}_i \Vdash_a X \ \textit{ iff } \ \forall i \in I, \mathfrak{F}_i \Vdash_a X.$$

That is, frame validity is a-invariant under a-disjoint unions.

Proof. $\biguplus_i \mathfrak{F}_i \Vdash_a X$ iff
for every model $\biguplus_i \mathfrak{M}_i = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ built on $\biguplus_i \mathfrak{F}_i$ and state $\mathfrak{s} \in \mathfrak{S}$, $v(\mathfrak{s}, a \supset X) = \top$ iff
for every model $\biguplus_i \mathfrak{M}_i = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ on $\biguplus_i \mathfrak{F}_i$, and $\mathfrak{s} \in \mathfrak{S}$, $a \leq v(\mathfrak{s}, X)$ iff
for every model $\biguplus_i \mathfrak{M}_i = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ on $\biguplus_i \mathfrak{F}_i$ and $\mathfrak{s} \in \mathfrak{S}$, $a \cdot v(\mathfrak{s}, X) = a$ iff
$\forall i \in I$, for every $\mathfrak{M}_i = \langle \mathfrak{S}_i, \mathfrak{g}_i, v_i \rangle$ built on $\mathfrak{F}_i$ and $\mathfrak{s} \in \mathfrak{S}_i$, $a \cdot v_i(\mathfrak{s}, X) = a$ iff
$\forall i \in I, \mathfrak{F}_i \Vdash_a X$. ∎

4.2 Generated Submodels-Subframes

We now take up the less trivial construction of generated submodels and subframes, but there still remains a warming-up style until we meet the final section with canonical extensions.

Definition 4.4 Let $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ and $\mathfrak{M}' = \langle \mathfrak{S}', \mathfrak{g}', v' \rangle$ be two $\mathcal{H}$-models for $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$. $\mathfrak{M}'$ is an *a-(preserving) $\mathcal{H}$-submodel* of $\mathfrak{M}$ if $\mathfrak{S}' \subseteq \mathfrak{S}$, $\mathfrak{g}'$ is the *a-restriction* of $\mathfrak{g}$ to $\mathfrak{S}' \times \mathfrak{S}'$, and v' is the *a-restriction* of v to $\mathfrak{S}' \times \Phi$:

1. for every $\mathfrak{s}, \mathfrak{t} \in \mathfrak{S}'$, $a \cdot \mathfrak{g}'(\mathfrak{s}, \mathfrak{t}) = a \cdot \mathfrak{g}(\mathfrak{s}, \mathfrak{t})$
2. for every $\mathfrak{s} \in \mathfrak{S}'$ and $p \in \Phi$, $a \cdot v'(\mathfrak{s}, p) = a \cdot v(\mathfrak{s}, p)$.

We call $\mathfrak{M}'$ *a-generated $\mathcal{H}$-submodel* of $\mathfrak{M}$ (and denote $\mathfrak{M}' \rightarrowtail^a \mathfrak{M}$) if $\mathfrak{M}'$ is a submodel of $\mathfrak{M}$ and for all states $\mathfrak{s}, \mathfrak{t} \in \mathfrak{S}$ the following *a-closure condition* holds:

if $\mathfrak{s} \in \mathfrak{S}'$ and $a \cdot \mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \neq \bot$, then $\mathfrak{t} \in \mathfrak{S}'$ $\qquad$ (a-cc)

For $\mathfrak{F} = \langle \mathfrak{S}, \mathfrak{g} \rangle$ and $\mathfrak{F}' = \langle \mathfrak{S}', \mathfrak{g}' \rangle$ $\mathcal{H}$-modal frames, $\mathfrak{F}'$ is called *a-subframe* of $\mathfrak{F}$ if $\mathfrak{g}'$ is the *a-restriction* of $\mathfrak{g}$ to $\mathfrak{S}' \times \mathfrak{S}'$, and *a-generated $\mathcal{H}$-subframe* (denoted $\mathfrak{F}' \rightarrowtail^a \mathfrak{F}$), if, moreover, the a-closure condition is satisfied.

Although it will not concern us later, it is plausible to quote the notion of a *submodel (subframe) generated by X*, which is the smallest submodel (subframe) of $\mathfrak{M}$ (of $\mathfrak{F}$) whose domain contains $X \subseteq \mathfrak{S}$.

Here is the invariance result in the model level:

Theorem 4.5 *Let $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ and $\mathfrak{M}' = \langle \mathfrak{S}', \mathfrak{g}', v' \rangle$ be two $\mathcal{H}$-models, such that $\mathfrak{M}'$ is an a-generated submodel of $\mathfrak{M}$. Then, for each formula A of $L^{\mathcal{H}}_{\Box\Diamond}$ and state $\mathfrak{s}$ of $\mathfrak{M}'$,*

$$a \cdot v(\mathfrak{s}, A) = a \cdot v'(\mathfrak{s}, A).$$

That is, modal satisfaction is a-invariant under a-generated submodels.

Proof. By induction on A (for all states of $\mathfrak{M}$ simultaneously):
For $A = p \in \Phi$ or $A = a \in \mathcal{H}$, trivially.
The $A = A_1 \wedge A_2$, $A_1 \vee A_2$ and $A_1 \supset A_2$ cases are just a routine and we prove them in detail for the last time.
For $A = A_1 \wedge A_2$,

$$\begin{aligned}
a \cdot v(\mathfrak{s}, A_1 \wedge A_2) &= a \cdot v(\mathfrak{s}, A_1) \cdot v(\mathfrak{s}, A_2) \\
&= \Big(a \cdot v(\mathfrak{s}, A_1)\Big) \cdot \Big(a \cdot v(\mathfrak{s}, A_2)\Big) \\
&= \Big(a \cdot v'(\mathfrak{s}, A_1)\Big) \cdot \Big(a \cdot v'(\mathfrak{s}, A_2)\Big) \qquad \text{(by Ind. Hyp.)} \\
&= a \cdot v'(\mathfrak{s}, A_1) \cdot v'(\mathfrak{s}, A_2) \\
&= a \cdot v'(\mathfrak{s}, A_1 \wedge A_2)
\end{aligned}$$

For $A = A_1 \vee A_2$,

$$\begin{aligned}
a \cdot v(\mathfrak{s}, A_1 \vee A_2) &= a \cdot \Big(v(\mathfrak{s}, A_1) + v(\mathfrak{s}, A_2)\Big) \\
&= \Big(a \cdot v(\mathfrak{s}, A_1)\Big) + \Big(a \cdot v(\mathfrak{s}, A_2)\Big) \\
&= \Big(a \cdot v'(\mathfrak{s}, A_1)\Big) + \Big(a \cdot v'(\mathfrak{s}, A_2)\Big) \qquad \text{(by Ind. Hyp.)} \\
&= a \cdot \Big(v'(\mathfrak{s}, A_1) + v'(\mathfrak{s}, A_2)\Big) \\
&= a \cdot v'(\mathfrak{s}, A_1 \vee A_2)
\end{aligned}$$

For $A = A_1 \supset A_2$,

$$\begin{aligned}
a \cdot v(\mathfrak{s}, A_1 \supset A_2) &= a \cdot \Big(v(\mathfrak{s}, A_1) \Rightarrow v(\mathfrak{s}, A_2) \Big) \\
&= a \cdot \bigg(\Big(a \cdot v(\mathfrak{s}, A_1) \Big) \Rightarrow \Big(a \cdot v(\mathfrak{s}, A_2) \Big) \bigg) && \text{(by } \mathcal{H}_{10}\text{)} \\
&= a \cdot \bigg(\Big(a \cdot v'(\mathfrak{s}, A_1) \Big) \Rightarrow \Big(a \cdot v'(\mathfrak{s}, A_2) \Big) \bigg) && \text{(by Ind. Hyp.)} \\
&= a \cdot \Big(v'(\mathfrak{s}, A_1) \Rightarrow v'(\mathfrak{s}, A_2) \Big) && (\mathcal{H}_{10}) \\
&= a \cdot v'(\mathfrak{s}, A_1 \supset A_2)
\end{aligned}$$

For $A = \Box A_1$, we have:

$$\begin{aligned}
a \cdot v(\mathfrak{s}, \Box A_1) &= a \cdot \prod_{\mathfrak{t} \in \mathfrak{S}} \Big(\mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \Rightarrow v(\mathfrak{t}, A_1) \Big) \\
&= \prod_{\mathfrak{t} \in \mathfrak{S}} \bigg(a \cdot \Big(\mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \Rightarrow v(\mathfrak{t}, A_1) \Big) \bigg) \\
&= \prod_{\mathfrak{t} \in \mathfrak{S}} a \cdot \Big(a \cdot \mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \Rightarrow a \cdot v(\mathfrak{t}, A_1) \Big) && \text{(by } \mathcal{H}_{10}\text{)} \\
&= \prod_{\mathfrak{t} \in \mathfrak{S}'} a \cdot \Big(a \cdot \mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \Rightarrow a \cdot v(\mathfrak{t}, A_1) \Big) && \text{(by } \mathcal{H}_1 \text{ \& Def. 4.4, } a\text{-cc)} \\
&= \prod_{\mathfrak{t} \in \mathfrak{S}'} a \cdot \Big(\underbrace{a \cdot \mathfrak{g}'(\mathfrak{s}, \mathfrak{t})}_{\text{by Def. 4.4}} \Rightarrow a \cdot v'(\mathfrak{t}, A_1) \Big) && \text{(by Ind. Hyp.)} \\
&= a \cdot \prod_{\mathfrak{t} \in \mathfrak{S}'} \Big(\mathfrak{g}'(\mathfrak{s}, \mathfrak{t}) \Rightarrow v'(\mathfrak{t}, A_1) \Big) \\
&= a \cdot v'(\mathfrak{s}, \Box A_1)
\end{aligned}$$

Finally, for $A = \Diamond A_1$, we take in a similar way:

$$\begin{array}{rcll}
a \cdot v(\mathfrak{s}, \Diamond A_1) & = & a \cdot \sum\limits_{\mathfrak{t}\,\in\,\mathfrak{S}} \Big(\mathfrak{g}(\mathfrak{s},\mathfrak{t}) \cdot v(\mathfrak{t}, A_1)\Big) & \\
& = & \sum\limits_{\mathfrak{t}\,\in\,\mathfrak{S}} \Big(a \cdot \mathfrak{g}(\mathfrak{s},\mathfrak{t}) \cdot v(\mathfrak{t}, A_1)\Big) & (\text{by } \mathcal{H}_1^+) \\
& = & \sum\limits_{\mathfrak{t}\,\in\,\mathfrak{S}} \Big(\Big(a \cdot \mathfrak{g}(\mathfrak{s},\mathfrak{t})\Big) \cdot \Big(a \cdot v(\mathfrak{t}, A_1)\Big)\Big) & \\
& = & \sum\limits_{\mathfrak{t}\,\in\,\mathfrak{S}'} \Big(\Big(a \cdot \mathfrak{g}(\mathfrak{s},\mathfrak{t})\Big) \cdot \Big(a \cdot v(\mathfrak{t}, A_1)\Big)\Big) & (\text{by } \mathcal{H}_1 \text{ \& Def. 4.4, } a\text{-cc}) \\
& = & \sum\limits_{\mathfrak{t}\,\in\,\mathfrak{S}'} \Big(\Big(\underbrace{a \cdot \mathfrak{g}'(\mathfrak{s},\mathfrak{t})}_{\text{by Def. 4.4}}\Big) \cdot \Big(a \cdot v'(\mathfrak{t}, A_1)\Big)\Big) & (\text{by Ind. Hyp.}) \\
& = & a \cdot \sum\limits_{\mathfrak{t}\,\in\,\mathfrak{S}'} \Big(\mathfrak{g}'(\mathfrak{s},\mathfrak{t}) \cdot v'(\mathfrak{t}, A_1)\Big) & \\
& = & a \cdot v'\Big(\mathfrak{s}, \Diamond A_1\Big) &
\end{array}$$

∎

Using the results we have proved so far, we can prove that various, naturally arising versions of the 'global' modality and the 'difference' modality are not definable using $\Box$ and $\Diamond$. The proof is very interesting but easy as well, and it is left to the reader.

Example 4.6 *None of the following logical modalities is definable in $L^{\mathcal{H}}_{\Box\Diamond}$: For every valuation v, state $\mathfrak{s}$, formula φ and value $a \in \mathcal{H}$,*

- $v(\mathfrak{s}, \Box_a \varphi) = \top$ *iff* $v(\mathfrak{t}, \varphi) = a$, *for every* $\mathfrak{t} \in \mathfrak{S}$
- $v(\mathfrak{s}, D_{a/b} \varphi) = \top$ *iff* $\exists \mathfrak{t} \in \mathfrak{S}$ *s.t.* $\mathfrak{t} \neq \mathfrak{s}$, $\mathfrak{g}(\mathfrak{s},\mathfrak{t}) = a$ *and* $v(\mathfrak{t}, \varphi) = b$
- $v(\mathfrak{s}, \Diamond_a \varphi) = \top$ *iff* $\exists \mathfrak{t} \in \mathfrak{S}$ *s.t.* $v(\mathfrak{t}, \varphi) = a$.

The frame validity preservation result has as follows:

Theorem 4.7 *Let $\mathfrak{F} = \langle \mathfrak{S}, \mathfrak{g} \rangle$ be an $\mathcal{H}$-frame. For every $X \in L^{\mathcal{H}}_{\Box\Diamond}$,*

$$\mathfrak{F} \Vdash_a X \quad \textit{iff for every } \mathcal{H}\textit{-frame } \mathfrak{F}',\ \mathfrak{F}' \rightarrowtail^a \mathfrak{F} \textit{ implies } \mathfrak{F}' \Vdash_a X.$$

That is, frame validity is a-invariant under a-generated subframes.

Proof. $\mathfrak{F} \Vdash_a X$ iff
for every model $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ built on $\mathfrak{F}$ and state $\mathfrak{s} \in \mathfrak{S}$, $v(\mathfrak{s}, a \supset X) = \top$ iff
for every model $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ built on $\mathfrak{F}$ and state $\mathfrak{s} \in \mathfrak{S}$, $a \cdot v(\mathfrak{s}, X) = a$ iff
for every $\mathfrak{M}' = \langle \mathfrak{S}', \mathfrak{g}', v' \rangle \rightarrowtail^a \mathfrak{M}$ built on $\mathfrak{F}' = \langle \mathfrak{S}', \mathfrak{g}' \rangle$ with $\mathfrak{F}' \rightarrowtail^a \mathfrak{F}$, and every state $\mathfrak{s} \in \mathfrak{S}'$, $a \cdot v'(\mathfrak{s}, X) = a$ iff
for every $\mathfrak{F}'$, $\mathfrak{F}' \rightarrowtail^a \mathfrak{F}$ implies $\mathfrak{F}' \Vdash_a X$. ∎

4.3 Bounded Morphisms

Definition 4.8 Let $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ and $\mathfrak{M}' = \langle \mathfrak{S}', \mathfrak{g}', v' \rangle$ be two $\mathcal{H}$-models for $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$. An *a-(preserving) bounded $\mathcal{H}$-morphism* from $\mathfrak{M}$ to $\mathfrak{M}'$ is a function $f : \mathfrak{S} \to^a \mathfrak{S}'$ satisfying the following conditions, where $\mathfrak{s}, \mathfrak{t} \in \mathfrak{S}$ are states of $\mathfrak{M}$ and $\mathrm{p} \in \Phi$ is a propositional variable:

1. $a \cdot v(\mathfrak{s}, p) = a \cdot v'(f(\mathfrak{s}), p)$
2. $a \cdot \mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \leq a \cdot \mathfrak{g}'(f(\mathfrak{s}), f(\mathfrak{t}))$ (forth)
3. for every $\mathfrak{s} \in \mathfrak{S}$ and $\mathfrak{u}' \in \mathfrak{S}'$ if $a \cdot \mathfrak{g}'(f(\mathfrak{s}), \mathfrak{u}') \neq \bot$, then there exists a $\mathfrak{u} \in \mathfrak{S}$ such that $a \cdot \mathfrak{g}(\mathfrak{s}, \mathfrak{u}) = a \cdot \mathfrak{g}'(f(\mathfrak{s}), \mathfrak{u}')$ & $f(\mathfrak{u}) = \mathfrak{u}'$ (back)

For $\mathfrak{F} = \langle \mathfrak{S}, \mathfrak{g} \rangle$ and $\mathfrak{F}' = \langle \mathfrak{S}', \mathfrak{g}' \rangle$ $\mathcal{H}$-modal frames, an *a-bounded $\mathcal{H}$-morphism from $\mathfrak{F}$ to $\mathfrak{F}'$* is a function $f : \mathfrak{S} \to \mathfrak{S}'$ which satisfies the above (forth) and (back) conditions.

If f is a *surjective* a-bounded $\mathcal{H}$-morphism from $\mathfrak{M}$ ($\mathfrak{F}$) to $\mathfrak{M}'$ ($\mathfrak{F}'$), we call $\mathfrak{M}'$ ($\mathfrak{F}'$) *a-bounded $\mathcal{H}$-morphic image of $\mathfrak{M}$ ($\mathfrak{F}$) through f* and we denote by $f : \mathfrak{M} \twoheadrightarrow^a \mathfrak{M}'$ ($\mathfrak{F} \twoheadrightarrow^a \mathfrak{F}'$). If there exists an f such that $f : \mathfrak{M} \twoheadrightarrow^a \mathfrak{M}'$ ($\mathfrak{F} \twoheadrightarrow^a \mathfrak{F}'$), we simply write $\mathfrak{M} \twoheadrightarrow^a \mathfrak{M}'$ ($\mathfrak{F} \twoheadrightarrow^a \mathfrak{F}'$).

We next present the invariance results in the model and frame levels.

Theorem 4.9 *Let $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ and $\mathfrak{M}' = \langle \mathfrak{S}', \mathfrak{g}', v' \rangle$ be two $\mathcal{H}$-models and $f : \mathfrak{S} \to^a \mathfrak{S}'$ an a-preserving bounded morphism. Then, for each formula A of $L^{\mathcal{H}}_{\Box\Diamond}$ and state $\mathfrak{s}$ of $\mathfrak{M}$,*

$$a \cdot v(\mathfrak{s}, A) \;=\; a \cdot v'(f(\mathfrak{s}), A).$$

That is, modal satisfaction is a-invariant under a-bounded morphisms.

Proof. By induction on A:
For atomic formulas, trivially.
We omit the routine cases of $A = A_1 \wedge A_2$ or $A_1 \vee A_2$ or $A_1 \supset A_2$, and proceed to the modal operators:

For $A = \Box A_1$, we split our proof into two parts:

($\geq$) $a \cdot v(\mathfrak{s}, \Box A_1) \;\geq\; a \cdot v'(f(\mathfrak{s}), \Box A_1)$.

$$\begin{array}{rcll}
a \cdot v(\mathfrak{s}, \Box A_1) & = & a \cdot \displaystyle\prod_{\mathfrak{t} \in \mathfrak{S}} \Big(\mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \Rightarrow v(\mathfrak{t}, A_1)\Big) & \\
& = & \displaystyle\prod_{\mathfrak{t} \in \mathfrak{S}} \Bigg(a \cdot \Big(\mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \Rightarrow v(\mathfrak{t}, A_1)\Big)\Bigg) & \\
& = & \displaystyle\prod_{\mathfrak{t} \in \mathfrak{S}} a \cdot \Big(a \cdot \mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \Rightarrow a \cdot v(\mathfrak{t}, A_1)\Big) & \text{(by } \mathcal{H}_{10}\text{)} \\
& \geq & \displaystyle\prod_{\mathfrak{t} \in \mathfrak{S}} a \cdot \Big(\underbrace{a \cdot \mathfrak{g}'(f(\mathfrak{s}), f(\mathfrak{t}))}_{\text{forth}} \Rightarrow a \cdot v'(f(\mathfrak{t}), A_1)\Big) & \text{(by Ind. Hyp. \& Def. 4.8)} \\
& = & \displaystyle\prod_{\mathfrak{t} \in \mathfrak{S}} a \cdot \Big(\mathfrak{g}'(f(\mathfrak{s}), f(\mathfrak{t})) \Rightarrow v'(f(\mathfrak{t}), A_1)\Big) & (\mathcal{H}_{10}) \\
& = & a \cdot \displaystyle\prod_{\mathfrak{t} \in \mathfrak{S}} \Big(\mathfrak{g}'(f(\mathfrak{s}), f(\mathfrak{t})) \Rightarrow v'(f(\mathfrak{t}), A_1)\Big) & \\
& \geq & a \cdot \displaystyle\prod_{\mathfrak{t}' \in \mathfrak{S}'} \Big(\mathfrak{g}'(f(\mathfrak{s}), \mathfrak{t}') \Rightarrow v'(\mathfrak{t}', A_1)\Big) & (f[\mathfrak{S}] \subseteq \mathfrak{S}') \\
& = & a \cdot v'(f(\mathfrak{s}), \Box A_1) & \text{(Def. 3.2)}
\end{array}$$

($\leq$) $a \cdot v'(f(\mathfrak{s}), \Box A_1) \;\geq\; a \cdot v(\mathfrak{s}, \Box A_1)$.

$$\begin{array}{rcll}
a \cdot v'(f(\mathfrak{s}), \Box A_1) & = & a \cdot \displaystyle\prod_{\mathfrak{t}' \in \mathfrak{S}'} \Big(\mathfrak{g}'(f(\mathfrak{s}), \mathfrak{t}') \Rightarrow v'(\mathfrak{t}', A_1)\Big) & \\
& = & \displaystyle\prod_{\mathfrak{t}' \in \mathfrak{S}'} a \cdot \Big(\mathfrak{g}'(f(\mathfrak{s}), \mathfrak{t}') \Rightarrow v'(\mathfrak{t}', A_1)\Big) & \\
& = & \displaystyle\prod_{\mathfrak{t}' \in \mathfrak{S}'} a \cdot \Big(a \cdot \mathfrak{g}'(f(\mathfrak{s}), \mathfrak{t}') \Rightarrow a \cdot v'(\mathfrak{t}', A_1)\Big) & (1)
\end{array}$$

Let $h : \mathfrak{S}' \to \mathfrak{S}$ be the function which, for every $\mathfrak{t}' \in \mathfrak{S}'$, picks one state $h(\mathfrak{t}') \in f^{-1}[\{\mathfrak{t}'\}] \subseteq \mathfrak{S}$, such that $a \cdot \mathfrak{g}'(f(\mathfrak{s}), \mathfrak{t}') = a \cdot \mathfrak{g}(\mathfrak{s}, h(\mathfrak{t}'))$. By the back condition (and the Axiom of Choice), such a function does exist and establishes the equality between the (1) and the following:

$$\begin{array}{rll}
& \displaystyle\prod_{\mathfrak{t}' \in \mathfrak{S}'} a \cdot \Big(\underbrace{a \cdot \mathfrak{g}(\mathfrak{s}, h(\mathfrak{t}'))}_{\textit{back}} \Rightarrow a \cdot v(h(\mathfrak{t}'), A_1)\Big) & \text{(by Ind. Hyp. \& Def. 4.8)} \\
= & a \cdot \displaystyle\prod_{\mathfrak{t}' \in \mathfrak{S}'} a \cdot \Big(a \cdot \mathfrak{g}(\mathfrak{s}, h(\mathfrak{t}')) \Rightarrow a \cdot v(h(\mathfrak{t}'), A_1)\Big) & \\
= & a \cdot \displaystyle\prod_{\mathfrak{t}' \in \mathfrak{S}'} \Big(\mathfrak{g}(\mathfrak{s}, h(\mathfrak{t}')) \Rightarrow v(h(\mathfrak{t}'), A_1)\Big) & \text{(by } \mathcal{H}_{10}\text{)} \\
\geq & a \cdot \displaystyle\prod_{\mathfrak{t} \in \mathfrak{S}} \Big(\mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \Rightarrow v(\mathfrak{t}, A_1)\Big) & (h[\mathfrak{S}'] \subseteq \mathfrak{S}) \\
= & a \cdot v(\mathfrak{s}, \Box A_1) &
\end{array}$$

For $A = \Diamond A_1$, again we prove the two inequalities:
$(\leq)$ $a \cdot v(\mathfrak{s}, \Diamond A_1) \ \leq \ a \cdot v'(f(\mathfrak{s}), \Diamond A_1)$.

$$
\begin{array}{rcll}
a \cdot v(\mathfrak{s}, \Diamond A_1) & = & a \cdot \sum\limits_{\mathfrak{t} \in \mathfrak{S}} \Big(\mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \cdot v(\mathfrak{t}, A_1) \Big) & \\
& = & \sum\limits_{\mathfrak{t} \in \mathfrak{S}} \Big(a \cdot \mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \cdot v(\mathfrak{t}, A_1) \Big) & \\
& = & \sum\limits_{\mathfrak{t} \in \mathfrak{S}} \Bigg(\Big(a \cdot \mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \Big) \cdot \Big(a \cdot v(\mathfrak{t}, A_1) \Big) \Bigg) & (\text{by } \mathcal{H}_1^+) \\
& \leq & \sum\limits_{\mathfrak{t} \in \mathfrak{S}} \Bigg(\Big(\underbrace{a \cdot \mathfrak{g}'(f(\mathfrak{s}), f(\mathfrak{t}))}_{forth} \Big) \cdot \Big(a \cdot v'(f(\mathfrak{t}), A_1) \Big) \Bigg) & (\text{by Ind. Hyp. \& Def. 4.8}) \\
& \leq & \sum\limits_{\mathfrak{t}' \in \mathfrak{S}'} \Bigg(\Big(a \cdot \mathfrak{g}'(f(\mathfrak{s}), \mathfrak{t}') \Big) \cdot \Big(a \cdot v'(\mathfrak{t}', A_1) \Big) \Bigg) & (\mathfrak{S} \subseteq \mathfrak{S}') \\
& = & \sum\limits_{\mathfrak{t}' \in \mathfrak{S}'} \Big(a \cdot \mathfrak{g}'(f(\mathfrak{s}), \mathfrak{t}') \cdot v'(\mathfrak{t}', A_1) \Big) & \\
& = & a \cdot \sum\limits_{\mathfrak{t}' \in \mathfrak{S}'} \Big(\mathfrak{g}'(f(\mathfrak{s}), \mathfrak{t}') \cdot v'(\mathfrak{t}', A_1) \Big) & (\mathcal{H}_1^+) \\
& = & a \cdot v' \Big(f(\mathfrak{s}), \Diamond A_1 \Big) &
\end{array}
$$

$(\geq)$ $a \cdot v'(f(\mathfrak{s}), \Diamond A_1) \ \leq \ a \cdot v(\mathfrak{s}, \Diamond A_1)$.

$$
\begin{array}{rcll}
a \cdot v'(f(\mathfrak{s}), \Diamond A_1) & = & \sum\limits_{\mathfrak{t}' \in \mathfrak{S}'} \Big(a \cdot \mathfrak{g}'(f(\mathfrak{s}), \mathfrak{t}') \cdot v'(\mathfrak{t}', A_1) \Big) & \\
& = & \sum\limits_{\mathfrak{t}' \in \mathfrak{S}'} \Bigg(\Big(a \cdot \mathfrak{g}'(f(\mathfrak{s}), \mathfrak{t}') \Big) \cdot \Big(a \cdot v'(\mathfrak{t}', A_1) \Big) \Bigg) & (2)
\end{array}
$$

Fix again an $h : \mathfrak{S}' \to \mathfrak{S}$ such that for every $\mathfrak{t}' \in \mathfrak{S}'$, $h(\mathfrak{t}') \in f^{-1}[\{\mathfrak{t}'\}]$, as in the proof of the $\Box A_{1(\leq)}$ case. The (2) is now equal to:

$$
\begin{array}{cll}
= & \sum\limits_{\mathfrak{t}' \in \mathfrak{S}'} \Bigg(\Big(\underbrace{a \cdot \mathfrak{g}(\mathfrak{s}, h(\mathfrak{t}'))}_{back} \Big) \cdot \Big(a \cdot v(h(\mathfrak{t}'), A_1) \Big) \Bigg) & (\text{by Ind. Hyp. \& Def. 4.8}) \\
\leq & \sum\limits_{\mathfrak{t} \in \mathfrak{S}} \Bigg(\Big(a \cdot \mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \Big) \cdot \Big(a \cdot v(\mathfrak{t}, A_1) \Big) \Bigg) & (h[\mathfrak{S}'] \subseteq \mathfrak{S}) \\
= & \sum\limits_{\mathfrak{t} \in \mathfrak{S}} \Big(a \cdot \mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \cdot v(\mathfrak{t}, A_1) \Big) & (\mathcal{H}_{10}) \\
= & a \cdot \sum\limits_{\mathfrak{t} \in \mathfrak{S}} \Big(\mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \cdot v(\mathfrak{t}, A_1) \Big) & (\mathcal{H}_1^+) \\
= & a \cdot v(\mathfrak{s}, \Diamond A_1) &
\end{array}
$$

∎

Theorem 4.10 *Let $\mathfrak{F} = \langle \mathfrak{S}, \mathfrak{g} \rangle$ and $\mathfrak{F}' = \langle \mathfrak{S}', \mathfrak{g}' \rangle$ be two $\mathcal{H}$-frames, and f a surjective a-bounded morphism : $\mathfrak{F} \twoheadrightarrow^a \mathfrak{F}'$. Then, for every $X \in L^{\mathcal{H}}_{\Box\Diamond}$,*

$$\mathfrak{F} \Vdash_a X \textit{ implies } \mathfrak{F}' \Vdash_a X.$$

That is, frame validity is a-invariant under a-bounded morphisms.

Proof. Let $\mathfrak{F} \Vdash_a X$. Then for every model $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ built on $\mathfrak{F}$ and state $\mathfrak{s} \in \mathfrak{S}$, $v(\mathfrak{s}, a \supset X) = \top$ or, $a \cdot v(\mathfrak{s}, X) = a$. If $\mathfrak{t}' \in \mathfrak{S}'$, there exists a $\mathfrak{t} \in \mathfrak{S}$ s.t. $f(\mathfrak{t}) = \mathfrak{t}'$. Thus, by theorem 4.9, if $\mathfrak{M}' = \langle \mathfrak{S}', \mathfrak{g}', v' \rangle$ is a model built on $\mathfrak{F}'$, $a \cdot v(\mathfrak{t}, X) = a \cdot v'(\mathfrak{t}', X)$ and, by the previous argument, $a \cdot v'(\mathfrak{t}', X) = a$ or $a \leq v'(\mathfrak{t}', X)$. Since $\mathfrak{M}'$ and $\mathfrak{t}'$ were arbitrarily chosen, we derive that $\mathfrak{F}' \Vdash_a X$. ∎

Here is an non-definability result concerning a certain class of $\mathcal{H}$-frames:

Example 4.11 *The class C of all $\mathcal{H}$-frames $\mathfrak{F} = \langle \mathfrak{S}, \mathfrak{g} \rangle$ s.t. $\forall \mathfrak{s} \big(\mathfrak{g}(\mathfrak{s}, \mathfrak{s}) = a \big)$ is not modally definable.*

Proof. Towards a contrary, suppose that there exists a formula $X \in L^{\mathcal{H}}_{\Box\Diamond}$ s.t. $\mathfrak{C} = \{ \mathfrak{F} : \mathfrak{F} \Vdash X \}$. Consider the $\mathcal{H}$-frame $\mathfrak{F}_1 = \langle \mathbb{N}, \mathfrak{g} \rangle$, where for every $n \in \mathbb{N}$, $\mathfrak{g}_1(n, n) = a$, $\mathfrak{g}_1(n, n+1) = \top$, and $\mathfrak{g}_1(m, n) = \bot$, for all others m, n. Then $\mathfrak{F}_1 \Vdash X$. Now, consider the $\mathcal{H}$-frame $\mathfrak{F}_2 = \langle 0, \mathfrak{g}_2 \rangle$, s.t. $\mathfrak{g}_2(0, 0) = \top$. It is easy to check that for the function $f : \mathfrak{F}_1 \to \mathfrak{F}_2$ with $f(n) = 0$ for every $n \in \mathbb{N}$ is a surjective $\top$-bounded morphism $: \mathfrak{F}_1 \twoheadrightarrow \mathfrak{F}_2$. However, $\mathfrak{F}_2 \nVdash X$, contradiction. ∎

Taking into into account that the class of $\mathcal{H}$-frames satisfying the condition $\forall \mathfrak{s} \big(\mathfrak{g}(\mathfrak{s}, \mathfrak{s}) \geq a \big)$ is modally definable by the formula $\mathbf{T}^a : \ \Box A \supset (a \supset A)$ ([14, Th. 4.6 (a)]), derive that the property $\forall \mathfrak{s} \big(\mathfrak{g}(\mathfrak{s}, \mathfrak{s}) \geq a \big)$ also cannot be modally definable.

There is an intimate relation between the notions of generated submodel (subframe) and bounded morphic image, and it is stated in the following theorem.

Theorem 4.12 *If f is an a-bounded $\mathcal{H}$-morphism from $\mathfrak{M}_0 = \langle \mathfrak{S}_0, \mathfrak{g}_0, v_0 \rangle$ to $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ (or from $\mathfrak{F}_0 = \langle \mathfrak{S}_0, \mathfrak{g}_0 \rangle$ to $\mathfrak{F} = \langle \mathfrak{S}, \mathfrak{g} \rangle$), then the a-bounded morphic image of $\mathfrak{M}_0$ ($\mathfrak{F}_0$) through f is an a-generated submodel (subframe) of $\mathfrak{M}$ ($\mathfrak{F}$).*

Proof. It suffices to work in the model level. Let $\mathfrak{M}' = \langle \mathfrak{S}', \mathfrak{g}', v' \rangle$ be the morphic image of $\mathfrak{M}_0$ through f, that is $f : \mathfrak{M}_0 \twoheadrightarrow^a \mathfrak{M}'$. Conditions 1 & 2 of definition 4.4 are trivially established by the stronger facts:

for every $\mathfrak{s}, \mathfrak{t} \in \mathfrak{S}'$, $\mathfrak{g}'(\mathfrak{s}, \mathfrak{t}) = \mathfrak{g}(\mathfrak{s}, \mathfrak{t})$

for every $\mathfrak{s} \in \mathfrak{S}$ and $p \in \Phi$, $v(\mathfrak{s}, p) = v'(\mathfrak{s}, p)$.

The argument for the a-closure condition goes as follows: Let $\mathfrak{s} \in \mathfrak{S}'$ and $a \cdot \mathfrak{g}'(\mathfrak{s}, \mathfrak{t}) \neq \bot$. Then there exists an $\mathfrak{s}_0 \in \mathfrak{S}_0$ s.t. $f(\mathfrak{s}_0) = \mathfrak{s}$. Thus, $a \cdot \mathfrak{g}'(f(\mathfrak{s}_0), \mathfrak{t}) \neq \bot$. By the back condition for the a-bounded morphism f, there exists a $\mathfrak{t}_0 \in \mathfrak{S}_0$, s.t. $a \cdot \mathfrak{g}_0(\mathfrak{s}_0, \mathfrak{t}_0) = a \cdot \mathfrak{g}'_0(\mathfrak{s}, \mathfrak{t})$ and $f(\mathfrak{t}_0) = \mathfrak{t}$. That is, $\mathfrak{t} \in \mathfrak{S}'$, which proves the a-closure condition for $\mathfrak{M}'$. ∎

4.4 Canonical Extensions

Let us first make a short reference to the canonical (or ultrafilter) extension construction in the two-valued modal logic, as it is presented in [2]. The interchangeable 'ultrafilter' in the title is due to the notions of ultrafilters which are involved there. To talk more explicitly, the ultrafilter extension $\mathfrak{ue}\mathfrak{F}$ of $\mathfrak{F}$ has as domain the set of all ultrafilters over the initial set of states $\mathfrak{S}$. What will be like the corresponding many-valued notion of an ultrafilter? The answer seems to emerge if we apply the 'functions instead of sets' principle (Remark 3.3). Indeed, considering each time the characteristic functions of sets, an ultrafilter over $\mathfrak{S}$ turns out to be a function from $2^{\mathfrak{S}}$ to 2 satisfying some properties. We point out that these properties are the ones of a homomorphism onto and, so, we take as set of states of our canonical extension to be (roughly) the set of all homomorphisms from $\mathcal{H}^{\mathfrak{S}}$ onto $\mathcal{H}$. Finally, to make these notions productive for our aim, a special refinement of them is included in the first section.

The route we follow through our invariance results is very similar to the two-valued case, but in proving those results, one automatically brings in his mind the completeness proof in [7]. Finally, the canonical construction requires the HA $\mathcal{H}$ be finite, and we explicitly mention it when we make use of this fact.

4.4.1 Basic definitions

In the two-valued case, the power set algebra $\langle P(\mathfrak{S}), \cap, \cup, ', \bot, \top\rangle$ is firstly formed and equipped with the two operators $\mathrm{m}_{\mathfrak{R}}$ and $\mathrm{l}_{\mathfrak{R}} : 2^{\mathfrak{S}} \to 2^{\mathfrak{S}}$:

$\mathrm{m}_{\mathfrak{R}}(X) = \{\mathfrak{s} \in \mathfrak{S} : \text{there exists } \mathfrak{t} \in X \text{ s.t. } \mathfrak{s}\,\mathfrak{R}\,\mathfrak{t}\}$ and

$\mathrm{l}_{\mathfrak{R}}(X) = \{\mathfrak{s} \in \mathfrak{S} : \text{for every } \mathfrak{t} \in \mathfrak{S}, \text{ if } \mathfrak{s}\,\mathfrak{R}\,\mathfrak{t} \text{ then } \mathfrak{t} \in X\}$.

It is later shown that the new *(full) complex algebra* formed is a *boolean algebra with operators*, sufficient to interpret all the propositonal modal operators, that is to provide an algebraic completeness result in respect to the proof system K, in terms of Ch. 5, [2]. However, the intended role of the complex algebra in the ultrafilter extension construction is to provide some of its subsets as states of the frame to be constructed.

We transfer all these to mv-ML, beginning with the corresponding notion of the complex algebra.

Definition 4.13 Let $\mathfrak{F} = \langle \mathfrak{S}, \mathfrak{g}\rangle$ be an $\mathcal{H}$-modal frame. We collect all functions (or sequences) which assign a truth value from $\mathcal{H}$ to each element of $\mathfrak{S}$, and convert it into *a Heyting algebra with operators*:

$$\mathfrak{F}^+ = \langle \mathcal{H}^{\mathfrak{S}}, +_{\mathfrak{F}^+}, \cdot_{\mathfrak{F}^+}, \Rightarrow_{\mathfrak{F}^+}, 0_{\mathfrak{F}^+}, 1_{\mathfrak{F}^+}, \mathrm{m}_{\mathfrak{g}}, \mathrm{l}_{\mathfrak{g}}\rangle.$$

$+_{\mathfrak{F}^+}, \cdot_{\mathfrak{F}^+}, \Rightarrow_{\mathfrak{F}^+}$ are defined coordinate-wisely with respect to the HA $\mathcal{H}$, that is, for example, if $\mathsf{f}, \mathsf{g} \in \mathcal{H}^{\mathfrak{S}}$ and $\mathfrak{s} \in \mathfrak{S}$, $(\mathsf{f} +_{\mathfrak{F}^+} \mathsf{g})(\mathfrak{s}) = \mathsf{f}(\mathfrak{s}) + \mathsf{g}(\mathfrak{s})$. $0_{\mathfrak{F}^+}, 1_{\mathfrak{F}^+}$ are the obvious constant functions and, finally, the operators $\mathrm{m}_{\mathfrak{g}}$ and $\mathrm{l}_{\mathfrak{g}} : \mathcal{H}^{\mathfrak{S}} \to \mathcal{H}^{\mathfrak{S}}$ are defined as follows: For every $\mathfrak{s} \in \mathfrak{S}$ and $\mathsf{f} \in \mathcal{H}^{\mathfrak{S}}$,

$$\mathrm{m}_{\mathfrak{g}}(\mathsf{f})(\mathfrak{s}) = \sum_{\mathfrak{t} \in \mathfrak{S}} \Big(\mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \cdot \mathsf{f}(\mathfrak{t})\Big)$$

$$\mathrm{l}_{\mathfrak{g}}(\mathsf{f})(\mathfrak{s}) = \prod_{\mathfrak{t} \in \mathfrak{S}} \Big(\mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \Rightarrow \mathsf{f}(\mathfrak{t})\Big).$$

Deviating again from the standard notation, we will mostly be writing $\mathfrak{F}^+$ instead of $\mathcal{H}^{\mathfrak{S}}$. Moreover, we keep the notation $\mathfrak{F}^+$ for the HA which is formed by dropping out the operators $\mathrm{m}_{\mathfrak{g}}$ and $\mathrm{l}_{\mathfrak{g}}$ from $\mathfrak{F}^+$. We also use $\prod_{\mathfrak{F}^+} T_0, \sum_{\mathfrak{F}^+} T_0$, for T_0 finite, to denote the obviously defined (finite) meets and joints of elements of $\mathfrak{F}^+$. To avoid being any superfluous, we should omit $1_{\mathfrak{F}^+}$ from the primitive operators of $\mathfrak{F}^+$, and define it as $1_{\mathfrak{F}^+} := 0_{\mathfrak{F}^+} \Rightarrow_{\mathfrak{F}^+} 1_{\mathfrak{F}^+}$, like in the definition of HAs 2.2. Finally, to carry on a more simplified presentation, we will mostly be omitting the subscripts in the operators of $\mathfrak{F}^+$, when they can be determined by the context.

We now turn to argue that this Heyting algebra with operators properly interprets the modal operators of mv-ML. But, of course, this cannot be meaningful, unless a proof system questioned to be (sound and) algebraically complete with our algebra is presented. To carry this on but without getting into irrelevant (to our main aim) tasks, we refer the reader to [7, Sec. 6] for a proof procedure, we claim here the algebraic completeness to be true, and we partly prove this in a coming lemma (4.15).

But first, we need one more definition. The reader may have already checked that the operators $\mathrm{m}_{\mathfrak{g}}$ and $\mathrm{l}_{\mathfrak{g}}$ fall respectively in $\mathrm{m}_{\mathfrak{R}}$ and $\mathrm{l}_{\mathfrak{R}}$, in case $\mathcal{H}$ is the boolean algebra 2. Another notion to be extended is the one of $V(\varphi)$, the set of states where φ is true in an $\mathcal{H}$-model $\mathfrak{M}$. Here, $[\varphi]_{\mathfrak{M}}$ will be that function which associates to each state $\mathfrak{s} \in \mathfrak{S}$ the truth value that φ 'has' in $\mathfrak{s}$, or $v(\mathfrak{s}, \varphi)$.

Definition 4.14 $[\varphi]_{\mathfrak{M}} : L^{\mathcal{H}}_{\Box\Diamond}(\Phi) \to \mathfrak{F}^+$ is defined to be the *truth function* which for every $\varphi \in L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$ and $\mathfrak{s} \in \mathfrak{S}$:

$$[\varphi]_{\mathfrak{M}}(\mathfrak{s}) = v(\mathfrak{s}, \varphi)$$

Now it can easily be checked that we have succeeded in the following shorthands:

$$[\Diamond\varphi]_{\mathfrak{M}} = \mathrm{m}_{\mathfrak{g}}([\varphi]_{\mathfrak{M}})$$

and

$$[\Box\varphi]_{\mathfrak{M}} = \mathrm{l}_{\mathfrak{g}}([\varphi]_{\mathfrak{M}}).$$

These relations constitute an evidence that 'the Heyting algebra with operators we have defined captures the modal notion of access between states in a natural way', [2, Sec. 5.2].

Before we proceed to the lemma which comprises some useful properties of $\mathrm{m}_{\mathfrak{g}}$ and $\mathrm{l}_{\mathfrak{g}}$ and partly indicates the algebraic completeness we announced as well, we pause for a remark concerning notation. According to our convention an element a of $\mathcal{H}$ is at the same time a formula of the modal language $L^{\mathcal{H}}_{\Box\Diamond}$. Hence, $[a]_{\mathfrak{M}}$ is supposed to denote the function which assigns to each state $\mathfrak{s}$ the value $v(\mathfrak{s}, a)$, which is, by definition 3.1, constantly a for every state $\mathfrak{s}$. Thus:

$$[a]_{\mathfrak{M}} \text{ is the constant function } \mathsf{l}_a \text{ of } \mathfrak{F}^+.$$

We will often omit the subscript '$\mathfrak{M}$' in $[\varphi]_{\mathfrak{M}}$, since it will only be used to represent truth functions based on $\mathfrak{M}$.

Lemma 4.15 *For every $a \in \mathcal{H}$ and $\mathsf{f}_1, \mathsf{f}_2 \in \mathfrak{F}^+$,*

1. *If $\mathsf{f}_1 \leq \mathsf{f}_2$, then $\mathrm{l}_{\mathfrak{g}}(\mathsf{f}_1) \leq \mathrm{l}_{\mathfrak{g}}(\mathsf{f}_2)$*
2. $\mathrm{l}_{\mathfrak{g}}(\mathsf{f}_1 \cdot_{\mathfrak{F}^+} \mathsf{f}_2) = \mathrm{l}_{\mathfrak{g}}(\mathsf{f}_1) \cdot_{\mathfrak{F}^+} \mathrm{l}_{\mathfrak{g}}(\mathsf{f}_2)$
3. $\mathrm{l}_{\mathfrak{g}}([a] \Rightarrow_{\mathfrak{F}^+} \mathsf{f}) = \Big([a] \Rightarrow_{\mathfrak{F}^+} \mathrm{l}_{\mathfrak{g}}(\mathsf{f})\Big)$
4. $\mathrm{l}_{\mathfrak{g}}(\mathsf{f} \Rightarrow_{\mathfrak{F}^+} [a]) = \Big(\mathrm{m}_{\mathfrak{g}}(\mathsf{f}) \Rightarrow_{\mathfrak{F}^+} [a]\Big)$

Proof. We proceed immediately to the proof of the second item, since the first one can then easily be derived by the following property which is satisfied by the random lattice $\mathcal{L}$: For every $a, b \in \mathcal{L}$, $a \leq b$ iff $a = a \cdot b$.

To prove the second, we have to show:

$$\text{For every } \mathfrak{s} \in \mathfrak{S},\ \mathrm{l}_{\mathfrak{g}}(\mathsf{f}_1 \cdot_{\mathfrak{F}^+} \mathsf{f}_2)(\mathfrak{s}) = \Big(\mathrm{l}_{\mathfrak{g}}(\mathsf{f}_1) \cdot_{\mathfrak{F}^+} \mathrm{l}_{\mathfrak{g}}(\mathsf{f}_2)\Big)(\mathfrak{s}).$$

$$\begin{aligned}
\Big(\mathrm{l}_{\mathfrak{g}}(\mathsf{f}_1) \cdot_{\mathfrak{F}^+} \mathrm{l}_{\mathfrak{g}}(\mathsf{f}_2)\Big)(\mathfrak{s}) &= \mathrm{l}_{\mathfrak{g}}(\mathsf{f}_1)(\mathfrak{s}) \cdot \mathrm{l}_{\mathfrak{g}}(\mathsf{f}_2)(\mathfrak{s}) && (\text{Def. of } \mathfrak{F}^+)\\
&= \prod_{\mathfrak{t} \in \mathfrak{S}} \Big(\mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \Rightarrow \mathsf{f}_1(\mathfrak{t})\Big) \cdot \prod_{\mathfrak{t} \in \mathfrak{S}} \Big(\mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \Rightarrow \mathsf{f}_2(\mathfrak{t})\Big)\\
&= \prod_{\mathfrak{t} \in \mathfrak{S}} \Big(\mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \Rightarrow \mathsf{f}_1(\mathfrak{t})\Big) \cdot \Big(\mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \Rightarrow \mathsf{f}_2(\mathfrak{t})\Big)\\
&= \prod_{\mathfrak{t} \in \mathfrak{S}} \Big(\mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \Rightarrow \mathsf{f}_1(\mathfrak{t}) \cdot \mathsf{f}_2(\mathfrak{t})\Big) && (\text{by } \mathcal{H}_6)\\
&= \mathrm{l}_{\mathfrak{g}}(\mathsf{f}_1 \cdot_{\mathfrak{F}^+} \mathsf{f}_2)(\mathfrak{s})
\end{aligned}$$

For 3, we prove:

$$\text{For every } \mathfrak{s} \in \mathfrak{S},\ \mathrm{l}_{\mathfrak{g}}([a] \Rightarrow \mathsf{f})(\mathfrak{s}) = \Big([a] \Rightarrow \mathrm{l}_{\mathfrak{g}}(\mathsf{f})\Big)(\mathfrak{s}).$$

$$
\begin{aligned}
\mathrm{l}_{\mathfrak{g}}([a] \Rightarrow \mathsf{f})(\mathfrak{s}) &= \prod_{\mathfrak{t}\,\in\,\mathfrak{S}} \Big(\mathfrak{g}(\mathfrak{s},\mathfrak{t}) \Rightarrow \big([a](\mathfrak{t}) \Rightarrow \mathsf{f}(\mathfrak{t})\big)\Big) \\
&= \prod_{\mathfrak{t}\,\in\,\mathfrak{S}} \Big(a \Rightarrow \big(\mathfrak{g}(\mathfrak{s},\mathfrak{t}) \Rightarrow \mathsf{f}(\mathfrak{t})\big)\Big) && \text{(by } \mathcal{H}_8\text{)} \\
&= a \Rightarrow \prod_{\mathfrak{t}\,\in\,\mathfrak{S}} \Big(\mathfrak{g}(\mathfrak{s},\mathfrak{t}) \Rightarrow \mathsf{f}(\mathfrak{t})\Big) && \text{(by } \mathcal{H}_1^{\cdot}\text{)} \\
&= [a](\mathfrak{s}) \Rightarrow \mathrm{l}_{\mathfrak{g}}(\mathsf{f})(\mathfrak{s}) \\
&= \Big([a] \Rightarrow \mathrm{l}_{\mathfrak{g}}(\mathsf{f})\Big)(\mathfrak{s})
\end{aligned}
$$

Similarly for 4:

$$
\begin{aligned}
\mathrm{l}_{\mathfrak{g}}(\mathsf{f} \Rightarrow [a])(\mathfrak{s}) &= \prod_{\mathfrak{t}\,\in\,\mathfrak{S}} \Big(\mathfrak{g}(\mathfrak{s},\mathfrak{t}) \Rightarrow \big(\mathsf{f}(\mathfrak{t}) \Rightarrow [a](\mathfrak{t})\big)\Big) \\
&= \prod_{\mathfrak{t}\,\in\,\mathfrak{S}} \Big(\mathfrak{g}(\mathfrak{s},\mathfrak{t}) \cdot \mathsf{f}(\mathfrak{t}) \Rightarrow a\Big) && \text{(by } \mathcal{H}_8\text{)} \\
&= \Big(\sum_{\mathfrak{t}\,\in\,\mathfrak{S}} \mathfrak{g}(\mathfrak{s},\mathfrak{t}) \cdot \mathsf{f}(\mathfrak{t})\Big) \Rightarrow a && \text{(by } \mathcal{H}_2^{\cdot}\text{)} \\
&= \mathrm{m}_{\mathfrak{g}}(\mathsf{f})(\mathfrak{s}) \Rightarrow [a](\mathfrak{s}) \\
&= \Big(\mathrm{m}_{\mathfrak{g}}(\mathsf{f}) \Rightarrow [a]\Big)(\mathfrak{s})
\end{aligned}
$$

∎

In fact, the part of the algebraic completeness that has been proved established through this lemma is that the operators $\mathrm{m}_{\mathfrak{g}}$ and $\mathrm{l}_{\mathfrak{g}}$ 'satisfy' or 'express' the modal axioms [7, p.13].

Canonical frame and model extensions. The final destination of the current section is to define the *canonical frame extension* and then, after determining what will be a valuation on that frame, the *canonical model extension*. The first thing we should do is to determine which elements will count as the states of their domain. The inspiration would be straightforward if something similar to the following fact held in the general case:

Fact 4.16 If F is an ultrafilter over a set S, then the characteristic function χ_F of F is a homomorphism from the powerset algebra with domain $P(S)$ onto the boolean algebra 2. And conversely, the inverse image of $\{1\}$ through a homomorphism from $P(S)$ onto 2 is an ultrafilter over S. That is, *there is an one-to-one correspondence between ultrafilters and onto homomorphisms.*

Although things become more complicated in the general case, we do begin our definitions motivated by the previous fact and explain in the sequence how things precisely are.

Definition 4.17 A homomorphism function h from $\mathfrak{F}^+$onto $\mathcal{H}$, which maps every $[a]_{\mathfrak{M}}$ to a, is called *good (over $\mathfrak{S}$)*.

The broader than the 0-1 domain of a HA urges on (or permits) a refinement of the previous definition:

Definition 4.18 Let T and A be respectively a subset and an element of $\mathfrak{F}^+$. A good function h is A-*good for/with respect to T* if for every finite subset T_0 of T,

$$\prod_{\mathsf{f}\,\in\,T_0} \mathsf{h}(\mathsf{f}) \not\leq \mathsf{h}(\mathsf{A}).$$

On the way to establish an analogous to the fact 4.16 result, we introduce the crucial notion of A-*finite intersection property (f.i.p.)* for a pair of subsets $\langle S, T\rangle$ of $\mathfrak{F}^+$: [2]

Definition 4.19 Let S, T and A be respectively two subsets and an element of $\mathfrak{F}^+$. We say that the pair $\langle S, T\rangle$ *has the* A-*f.i.p.* if for every finite subsets $S_0 \subseteq S$ and $T_0 \subseteq T$, there exists a good function h s.t.:

- $\mathsf{h}[S_0] = \{\top\}$
- $\prod_{\mathsf{f}\,\in\,T_0} \mathsf{h}(\mathsf{f}) \not\leq \mathsf{h}(\mathsf{A})$.

We say that $\langle S, T\rangle$ *maximally has the* A-*f.i.p.* if $S \neq \emptyset$, $\langle S, T\rangle$ has the A-f.i.p. and, for every S' containing S s.t. $\langle S', T\rangle$ has the A-f.i.p., $S = S'$.

Note that if $\langle S, T\rangle$ maximally has the A-f.i.p., then $[\top]_{\mathfrak{M}} = 1_{\mathfrak{F}^+} \in S$.

Definition 4.19 would be stronger if we demanded: for every finite subset S_0 of S, there exists an A-good function h for T, which maps every element of S_0 to $\top$ ($\mathsf{h}[S_0] = \{\top\}$). This version is not much suitable for us here but it is the one which is stated in theorem 4.22 to establish the analogy with the fact 4.16. But let us first argue about the existence of maximal sets with the A-f.i.p. and their essential property.

Proposition 4.20 *If $\langle S, T\rangle$ has the A-f.i.p., then it can be extended to a maximal $\langle \Delta, T\rangle$ with the A-f.i.p. Moreover, it holds that for every $\mathsf{f} \in \mathfrak{F}^+$, there exists a unique $t \in \mathcal{H}$ such that $\big(\mathsf{f} \Leftrightarrow [t]\big) \in \Delta$, where recall (Lemma 2.3) that $\big(\mathsf{f} \Leftrightarrow [t]\big)$ is an abbreviation of the function $\big(\mathsf{f} \Rightarrow [t]\big) \cdot_{\mathfrak{F}^+} \big([t] \Rightarrow \mathsf{f}\big)$ of $\mathfrak{F}^+$.*

[2]Likely the completeness proof in [7], it is impossible to avoid the pair $\langle S, T\rangle$ and work only with its second component.

Proof. We will produce the set Δ by a typical appealing to Zorn's lemma. Let $\langle S, T\rangle$ have the A-f.i.p. and form the set $\Gamma = \{G : S \subseteq G \;\&\; \langle G, T\rangle$ has the A-f.i.p.$\}$. Γ is not empty, since $S \in \Gamma$. Let $\{G_i\}_{i\in I}$ be a chain of elements of Γ. Then, if $H = \bigcup\{G_i : i \in I\}$ belongs to Γ, it will be an upper bound for $\{G_i\}_{i\in I}$ in Γ, and hence, by Zorn's lemma, Γ will have maximal elements.

$H \supseteq S$ and hence, to prove that $H \in \Gamma$, it suffices to show that $\langle H, T\rangle$ has the A-f.i.p. Take H_0 and T_0 be finite subsets of H and T respectively. Since H_0 is finite and $\{G_i\}_{i\in I}$ is a chain, there exists a $k \in I$ s.t. $H_0 \subseteq G_k$. $G_k \in \Gamma$, thus there exists a good function h s.t. $\mathsf{h}[S_0] = \{\top\}$ and $\prod\limits_{\mathsf{f}\,\in\, T_0} \mathsf{h}(\mathsf{f}) \not\leq \mathsf{h}(\mathsf{A})$. This h establishes that $\langle H, T\rangle$ has the A-f.i.p.

We now proceed to the second part of the proposition and we prove it through the following three claims for the maximal $\langle \Delta, T\rangle$ with the A-f.i.p.:

Claim 1: For every $\mathsf{f}, \mathsf{g} \in \mathfrak{F}^+$, if $\mathsf{f} \in \Delta$ and $\mathsf{f} \leq \mathsf{g}$, then $\mathsf{g} \in \Delta$.

Indeed, if g was not in Δ then $\langle \Delta \cup \{\mathsf{g}\}, T\rangle$ should not have the A-f.i.p. Thus, there would exist finite subsets $\Delta_0 \subseteq \Delta$ and $T_0 \subseteq T$ s.t. for every good function h, if $\mathsf{h}[\Delta_0 \cup \{\mathsf{g}\}] = \{\top\}$, $\mathsf{h}(\prod T_0) \leq \mathsf{h}(\mathsf{A})$. But then $\langle \Delta, T\rangle$ would not have the A-f.i.p., because for the finite subsets $\Delta_0 \cup \{\mathsf{f}\} \subseteq \Delta$ and $T_0 \subseteq T$, every good h with $\mathsf{h}[\Delta_0 \cup \{\mathsf{f}\}] = \{\top\}$, would satisfy $\mathsf{h}[\Delta_0 \cup \{\mathsf{g}\}] = \{\top\}$, and hence, $\mathsf{h}(\prod T_0) \leq \mathsf{h}(\mathsf{A})$.

Claim 2: For every $\mathsf{f} \in \mathfrak{F}^+$, there exists a $t \in \mathcal{H}$, s.t. $\big(\mathsf{f} \Leftrightarrow [t]\big) \in \Delta$.

We assume here the finiteness of $\mathcal{H} = \{t_1, \ldots, t_n\}$. Towards a contrary, suppose that there exists an $\mathsf{f} \in \mathfrak{F}^+$ s.t. for every $i \in \{1, \ldots, n\}$, $\big(\mathsf{f} \Leftrightarrow [t_i]\big) \notin \Delta$. Hence, for every $i \in \{1, \ldots, n\}$, $\langle \Delta \cup \{\mathsf{f} \Leftrightarrow [t_i]\}, T\rangle$ has not the A-f.i.p., or, for every $i \in \{1, \ldots, n\}$, there exist finite $\Delta_i \subseteq \Delta$ and $T_i \subseteq T$, s.t. for every good h, if $\mathsf{h}[\Delta_i \cup \{\mathsf{f} \Leftrightarrow [t_i]\}] = \{\top\}$, then $\mathsf{h}(\prod T_i) \leq \mathsf{h}(\mathsf{A})$. Let $\Delta_0 = \Delta_1 \cup \ldots \cup \Delta_n$ and $T_0 = T_1 \cup \ldots \cup T_n$. Then, for every good h, if $\mathsf{h}[\Delta_0 \cup \{\mathsf{f} \Leftrightarrow [t_i]\}] = \{\top\}$, $\mathsf{h}(\prod T_0) \leq \mathsf{h}(\mathsf{A})$. Finally, it is now derived that for the finite subsets $\Delta_0 \subseteq \Delta$ and $T_0 \subseteq T$, if for a good h, $\mathsf{h}[\Delta_0] = \{\top\}$, then since $\mathsf{h}(\mathsf{f}) = t_k$ for some $k \in \{1, \ldots, n\}$, $\mathsf{h}[\Delta_0 \cup \{\mathsf{f} \Leftrightarrow [t_k]\}] = \{\top\}$, whence $\mathsf{h}(\prod T_0) \leq \mathsf{h}(\mathsf{A})$, which contradicts to the fact that $\langle \Delta, T\rangle$ has A-f.i.p.

Claim 3: For every $\mathsf{f}, \mathsf{g} \in \Delta$, if $\mathsf{f}, \mathsf{g} \in \Delta$, then $\mathsf{f} \cdot_{\mathfrak{F}^+} \mathsf{g} \in \Delta$.

To a contrary, suppose $\mathsf{f} \cdot_{\mathfrak{F}^+} \mathsf{g} \notin \Delta$. Then, $\big(\mathsf{f} \cdot_{\mathfrak{F}^+} \mathsf{g} \Leftrightarrow [\top]\big) \notin \Delta$, because $\mathsf{f} \cdot_{\mathfrak{F}^+} \mathsf{g} = \big(\mathsf{f} \cdot_{\mathfrak{F}^+} \mathsf{g} \Leftrightarrow [\top]\big)$ (by lemma 2.3 (1)). Now, by Claim 2, there exists a $t \in \mathcal{H}$, with $t \neq \top$ and $\big(\mathsf{f} \cdot_{\mathfrak{F}^+} \mathsf{g} \Leftrightarrow [t]\big) \in \Delta$. By Claim 1, $\big(\mathsf{f} \cdot_{\mathfrak{F}^+} \mathsf{g} \Rightarrow [t]\big) \in \Delta$. Now, we can reach a contradiction by proving that $\langle \Delta, T\rangle$ has not the A-f.i.p.: Take $D_0 = \{\mathsf{f}, \mathsf{g}, \big(\mathsf{f} \cdot_{\mathfrak{F}^+} \mathsf{g} \Rightarrow [t]\big)\}$ and T_0 a random finite subset of T. Then, no good function h can give $\mathsf{h}[\Delta_0] = \{\top\}$, because otherwise, $\top = \mathsf{h}(\mathsf{f}) \cdot \mathsf{h}(\mathsf{g}) \cdot \mathsf{h}(\mathsf{f} \cdot_{\mathfrak{F}^+} \mathsf{g} \Rightarrow [t]) = \mathsf{h}\big(\mathsf{f} \cdot_{\mathfrak{F}^+} \mathsf{g} \cdot_{\mathfrak{F}^+} (\mathsf{f} \cdot_{\mathfrak{F}^+} \mathsf{g} \Rightarrow [t])\big) \leq \mathsf{h}([t]) = t$. Thus, by a trivial reasoning, for every good function h, if $\mathsf{h}[\Delta_0] = \{\top\}$, $\mathsf{h}(\prod T_0) \leq \mathsf{h}(\mathsf{A})$, whence $\langle \Delta, T\rangle$ has not the A-f.i.p.

Claim 3 is used to prove that there can not exist two different members $t_1, t_2 \in \mathcal{H}$, s.t. for an $\mathsf{f} \in \mathfrak{F}^+$ both $\mathsf{f} \Leftrightarrow [t_1]$ and $\mathsf{f} \Leftrightarrow [t_2]$ are in Δ. For otherwise, combining Claims 1 & 3, $[t_1] \Leftrightarrow [t_2]$ should be in Δ, but for every good function h, $\mathsf{h}([t_1] \Leftrightarrow [t_2]) \neq \top$, since $t_1 \neq t_2$, and hence by a trivial reasoning, $\langle \Delta, T \rangle$ would have not have the A-f.i.p. ∎

Remark 4.21 Claims 1 and 3 of previous proposition provide properties which establish $\langle \Delta, T \rangle$ (equivalently) be an *A-ultrafilter over* $\mathfrak{F}^+$ once, of course, such a notion is defined. We will not go into this; we just state in the next theorem the correspondence between maximal pairs having the A-f.i.p. and A-good functions, as we have promised.

Theorem 4.22 *If $\langle \Delta, T \rangle$ maximally has the A-f.i.p., then the function $\mathsf{h}_\vartriangle : \mathcal{H}^{\mathfrak{S}} \to \mathcal{H}$ described by:*
for every $\mathsf{f} \in \mathfrak{F}^+$,

$$\mathsf{h}_\vartriangle(\mathsf{f}) = \textit{the unique } t \in \mathcal{H} \textit{ s.t. } \big(\mathsf{f} \Leftrightarrow [t]\big) \in \Delta,$$

is A-good with respect to T, with $\mathsf{h}_\vartriangle[\Delta] = \{\top\}$.

Conversely, if h is an A-good function with respect to a set $T \subseteq \mathfrak{F}^+$, then there exists a (unique) set Δ s.t. $\langle \Delta, T \rangle$ maximally has the A-f.i.p. and $\mathsf{h} = \mathsf{h}_\vartriangle$.

PROOF. Note that during this proof we (will) make use of a lemma which is next proved.

For the first part, we show that $\mathsf{h}_\vartriangle$ is a good function, i.e. a homomorphism from $\mathcal{H}^{\mathfrak{S}}$ onto $\mathcal{H}$ mapping each $[a]_{\mathfrak{M}}$ to a. Let $\triangle$ represent any of $\cdot$, $+$, $\Rightarrow$ and $\triangle_{\mathfrak{F}^+}$ the corresponding $+_{\mathfrak{F}^+}, \cdot_{\mathfrak{F}^+}, \Rightarrow_{\mathfrak{F}^+}$ of $\mathfrak{F}^+$. Then, for every f_1 and f_2 in $\mathfrak{F}^+$, there are unique t_1 and t_2 in $\mathcal{H}$, s.t. $\big(\mathsf{f}_1 \Leftrightarrow [t_1]\big), \big(\mathsf{f}_2 \Leftrightarrow [t_2]\big) \in \Delta$, where from we derive according lemma 4.23: $\big(\mathsf{f}_1 \triangle_{\mathfrak{F}^+} \mathsf{f}_2 \Leftrightarrow [t_1 \triangle t_2]\big) \in \Delta$, and hence, $\mathsf{h}_\vartriangle(\mathsf{f}_1 \triangle_{\mathfrak{F}^+} \mathsf{f}_2) = t_1 \triangle t_2 = \mathsf{h}_\vartriangle(\mathsf{f}_1) \triangle \mathsf{h}_\vartriangle(\mathsf{f}_2)$.

Also, for every $a \in \mathcal{H}, \mathsf{h}([a]_{\mathfrak{M}}) = a$, since $\big([a] \Leftrightarrow [a]\big) = [\top] \in \Delta$.

That $\mathsf{h}_\vartriangle$ is A-good for T is easy: to a contrary, assume that for a finite subset T_0 of T, $\prod_{\mathsf{f} \in T_0} \mathsf{h}_\vartriangle(\mathsf{f}) \leq \mathsf{h}_\vartriangle(\mathsf{A})$. Then, $\mathsf{h}_\vartriangle(\prod T_0 \Rightarrow_{\mathfrak{F}^+} \mathsf{A}) = \top$, or $(\prod T_0 \Rightarrow_{\mathfrak{F}^+} \mathsf{A}) \in \Delta$. This leads to the contradiction that $\langle \Delta, T \rangle$ has not the A-f.i.p.: for the finite sets $\Delta_0 = \{\prod T_0 \Rightarrow_{\mathfrak{F}^+} \mathsf{A})\}$ and T_0, every h s.t. $\mathsf{h}[\Delta_0] = \{\top\}$ satisfies $\mathsf{h}(\prod T_0) \leq \mathsf{h}(\mathsf{A})$.

Finally, $\mathsf{h}_\vartriangle[\Delta] = \{\top\}$, since for every $\mathsf{f} \in \Delta$, $\big(\mathsf{f} \Leftrightarrow [\top]\big) \in \Delta$.

For the inverse, if h is an A-good function with respect to $T \subseteq \mathfrak{F}^+$, then the desired Δ is the set $\mathsf{h}^{-1}[\{\top\}] = \{\mathsf{f} \in \mathfrak{F}^+ : \mathsf{h}(\mathsf{f}) = \top\}$. Indeed,

- $\langle \Delta, T \rangle$ has the A-f.i.p.: For every finite subsets $\Delta_0 \subseteq \Delta$ and $T_0 \subseteq T$, $\mathsf{h}[\Delta_0] = \{\top\}$ by the definition of Δ and $\prod_{\mathsf{f} \in T_0} \mathsf{h}(\mathsf{f}) \not\leq \mathsf{h}(\mathsf{A})$ since h is A-good function for T.

- $\langle \Delta, T \rangle$ maximally has the A-f.i.p.: Extend $\langle \Delta, T \rangle$ to a maximal $\langle \Delta', T \rangle$ with the A-f.i.p., according to proposition 4.20. We show $\Delta = \Delta'$. Let $\mathsf{f} \in \Delta'$. Then, by lemma 2.3 (1), $(\mathsf{f} \Leftrightarrow [\top]) \in \Delta'$. Now, since h is a function, there exists a $t \in \mathcal{H}$ s.t. $\mathsf{h}(\mathsf{f}) = t$. Hence, $\mathsf{h}\big(\mathsf{f} \Leftrightarrow [t]\big) = \top$, which implies $\big(\mathsf{f} \Leftrightarrow [t]\big) \in \Delta \subseteq \Delta'$, and this forces t be $\top$. That is, $\mathsf{h}(\mathsf{f}) = \top$ and hence $\mathsf{f} \in \Delta$.

- $\mathsf{h}_\triangle = \mathsf{h}$: $\mathsf{h}_\triangle(\mathsf{f}) = t$ iff $\big(\mathsf{f} \Leftrightarrow [t]\big) \in \Delta$ iff $\mathsf{h}\big(\mathsf{f} \Leftrightarrow [t]\big) = \top$ iff $\mathsf{h}(\mathsf{f}) = t$.

It should now be obvious that this Δ is the unique set s.t. $\langle \Delta, T\rangle$ maximally has the A-f.i.p. and $\mathsf{h} = \mathsf{h}_\triangle$. ∎

Lemma 4.23 *If* $\big(\mathsf{f}_1 \Leftrightarrow [t_1]\big), \big(\mathsf{f}_2 \Leftrightarrow [t_2]\big) \in \Delta$, *then* $\big(\mathsf{f}_1 \triangle_{\mathfrak{F}^+} \mathsf{f}_2 \Leftrightarrow [t_1 \triangle t_2]\big) \in \Delta$.

Proof. Since $[t_1 \triangle t_2] = [t_1] \triangle_{\mathfrak{F}^+} [t_2]$, it suffices to show that for every $\mathsf{f}_1, \mathsf{f}_2, \mathsf{g}_1, \mathsf{g}_2 \in \mathfrak{F}^+$ the following three hold:
(1) if $(\mathsf{f}_1 \Leftrightarrow \mathsf{g}_1), (\mathsf{f}_2 \Leftrightarrow \mathsf{g}_2) \in \Delta$, then $(\mathsf{f}_1 \cdot_{\mathfrak{F}^+} \mathsf{f}_2 \Leftrightarrow \mathsf{g}_1 \cdot_{\mathfrak{F}^+} \mathsf{g}_2) \in \Delta$.
$(\mathsf{f}_1 \Leftrightarrow \mathsf{g}_1) = (\mathsf{f}_1 \Rightarrow \mathsf{g}_1) \cdot (\mathsf{g}_1 \Rightarrow \mathsf{f}_1) \leq (\mathsf{f}_1 \Rightarrow \mathsf{g}_1)$ and $(\mathsf{f}_2 \Leftrightarrow \mathsf{g}_2) = (\mathsf{f}_2 \Rightarrow \mathsf{g}_2) \cdot (\mathsf{g}_2 \Rightarrow \mathsf{f}_2) \leq (\mathsf{f}_2 \Rightarrow \mathsf{g}_2)$, hence by Claim 1 of proposition 4.20, $(\mathsf{f}_1 \Rightarrow \mathsf{g}_1), (\mathsf{f}_2 \Rightarrow \mathsf{g}_2) \in \Delta$. Moreover, by property $\mathcal{H}_4$, $(\mathsf{f}_1 \Rightarrow \mathsf{g}_1) \leq (\mathsf{f}_1 \cdot \mathsf{f}_2 \Rightarrow \mathsf{g}_1)$ and $(\mathsf{f}_2 \Rightarrow \mathsf{g}_2) \leq (\mathsf{f}_1 \cdot \mathsf{f}_2 \Rightarrow \mathsf{g}_2)$, thus $(\mathsf{f}_1 \cdot \mathsf{f}_2 \Rightarrow \mathsf{g}_1) \in \Delta$ and $(\mathsf{f}_1 \cdot \mathsf{f}_2 \Rightarrow \mathsf{g}_2) \in \Delta$. By Claim 3 of prop. 4.20, $(\mathsf{f}_1 \cdot \mathsf{f}_2 \Rightarrow \mathsf{g}_1) \cdot (\mathsf{f}_1 \cdot \mathsf{f}_2 \Rightarrow \mathsf{g}_2) \in \Delta$. But, $(\mathsf{f}_1 \cdot \mathsf{f}_2 \Rightarrow \mathsf{g}_1) \cdot (\mathsf{f}_1 \cdot \mathsf{f}_2 \Rightarrow \mathsf{g}_2) \leq (\mathsf{f}_1 \cdot \mathsf{f}_2 \Rightarrow \mathsf{g}_1 \cdot \mathsf{g}_2)$ (by $\mathcal{H}_6$) and hence $(\mathsf{f}_1 \cdot \mathsf{f}_2 \Rightarrow \mathsf{g}_1 \cdot \mathsf{g}_2) \in \Delta$. By a similar argument we can prove $(\mathsf{g}_1 \cdot \mathsf{g}_2 \Rightarrow \mathsf{f}_1 \cdot \mathsf{f}_2) \in \Delta$, thus $(\mathsf{f}_1 \cdot_{\mathfrak{F}^+} \mathsf{f}_2 \Leftrightarrow \mathsf{g}_1 \cdot_{\mathfrak{F}^+} \mathsf{g}_2) = (\mathsf{f}_1 \cdot \mathsf{f}_2 \Rightarrow \mathsf{g}_1 \cdot \mathsf{g}_2) \cdot (\mathsf{g}_1 \cdot \mathsf{g}_2 \Rightarrow \mathsf{f}_1 \cdot \mathsf{f}_2) \in \Delta$.
(2) if $(\mathsf{f}_1 \Leftrightarrow \mathsf{g}_1), (\mathsf{f}_2 \Leftrightarrow \mathsf{g}_2) \in \Delta$, then $(\mathsf{f}_1 +_{\mathfrak{F}^+} \mathsf{f}_2 \Leftrightarrow \mathsf{g}_1 +_{\mathfrak{F}^+} \mathsf{g}_2) \in \Delta$.
Again, we have $(\mathsf{f}_1 \Rightarrow \mathsf{g}_1), (\mathsf{f}_2 \Rightarrow \mathsf{g}_2) \in \Delta$. Now, by $\mathcal{H}_5$, $(\mathsf{f}_1 \Rightarrow \mathsf{g}_1) \leq (\mathsf{f}_1 \Rightarrow \mathsf{g}_1 + \mathsf{g}_2)$ and $(\mathsf{f}_2 \Rightarrow \mathsf{g}_2) \leq (\mathsf{f}_2 \Rightarrow \mathsf{g}_1 + \mathsf{g}_2)$, thus $(\mathsf{f}_1 \Rightarrow \mathsf{g}_1 + \mathsf{g}_2) \in \Delta$ and $(\mathsf{f}_2 \Rightarrow \mathsf{g}_1 + \mathsf{g}_2) \in \Delta$. Hence, $(\mathsf{f}_1 \Rightarrow \mathsf{g}_1 + \mathsf{g}_2) \cdot (\mathsf{f}_2 \Rightarrow \mathsf{g}_1 + \mathsf{g}_2) \in \Delta$. But, $(\mathsf{f}_1 \Rightarrow \mathsf{g}_1 + \mathsf{g}_2) \cdot (\mathsf{f}_2 \Rightarrow \mathsf{g}_1 + \mathsf{g}_2) \leq (\mathsf{f}_1 + \mathsf{f}_2 \Rightarrow \mathsf{g}_1 + \mathsf{g}_2)$ (by $\mathcal{H}_7$), which implies $(\mathsf{f}_1 + \mathsf{f}_2 \Rightarrow \mathsf{g}_1 + \mathsf{g}_2) \in \Delta$. Similarly, $(\mathsf{g}_1 + \mathsf{g}_2 \Rightarrow \mathsf{f}_1 + \mathsf{f}_2) \in \Delta$, hence $(\mathsf{f}_1 +_{\mathfrak{F}^+} \mathsf{f}_2 \Leftrightarrow \mathsf{g}_1 +_{\mathfrak{F}^+} \mathsf{g}_2) \in \Delta$.
(3) if $(\mathsf{f}_1 \Leftrightarrow \mathsf{g}_1), (\mathsf{f}_2 \Leftrightarrow \mathsf{g}_2) \in \Delta$, then $(\mathsf{f}_1 \Rightarrow_{\mathfrak{F}^+} \mathsf{f}_2) \Leftrightarrow (\mathsf{g}_1 \Rightarrow_{\mathfrak{F}^+} \mathsf{g}_2) \in \Delta$.
We have $(\mathsf{f}_1 \Leftrightarrow \mathsf{g}_1) \leq (\mathsf{g}_1 \Rightarrow \mathsf{f}_1)$ and $(\mathsf{f}_2 \Leftrightarrow \mathsf{g}_2) \leq (\mathsf{f}_2 \Rightarrow \mathsf{g}_2)$, hence $(\mathsf{g}_1 \Rightarrow \mathsf{f}_1) \cdot (\mathsf{f}_2 \Rightarrow \mathsf{g}_2) \in \Delta$. We argue as follows: $\mathsf{g}_1 \cdot (\mathsf{g}_1 \Rightarrow \mathsf{f}_1) \cdot (\mathsf{f}_1 \Rightarrow \mathsf{f}_2) \cdot (\mathsf{f}_2 \Rightarrow \mathsf{g}_2) \leq \mathsf{g}_2$ ($\mathcal{H}_1$ & $\mathcal{H}_9$). Equivalently, applying $\mathcal{H}_1$ two times, $(\mathsf{g}_1 \Rightarrow \mathsf{f}_1) \cdot (\mathsf{f}_2 \Rightarrow \mathsf{g}_2) \leq (\mathsf{f}_1 \Rightarrow \mathsf{f}_2) \Rightarrow (\mathsf{g}_1 \Rightarrow \mathsf{g}_2)$. Thus, $(\mathsf{f}_1 \Rightarrow \mathsf{f}_2) \Rightarrow (\mathsf{g}_1 \Rightarrow \mathsf{g}_2) \in \Delta$. Similarly, we can prove $(\mathsf{g}_1 \Rightarrow \mathsf{g}_2) \Rightarrow (\mathsf{f}_1 \Rightarrow \mathsf{f}_2) \in \Delta$, hence $(\mathsf{f}_1 \Rightarrow_{\mathfrak{F}^+} \mathsf{f}_2 \Leftrightarrow \mathsf{g}_1 \Rightarrow_{\mathfrak{F}^+} \mathsf{g}_2) \in \Delta$. ∎

Remark 4.24 Since for every $\mathsf{f} \in \mathfrak{F}^+, \mathsf{f} = (\mathsf{f} \Leftrightarrow 1_{\mathfrak{F}^+})$ (lemma 2.3 (1)), it is direct from the definition of $\mathsf{h}_\triangle$ that for every $\mathsf{f} \in \mathfrak{F}^+, \mathsf{f} \in \Delta$ iff $\mathsf{h}_\triangle(\mathsf{f}) = \top$.
Let also be noted that the discussion about A-goodness (leading to theorem 4.22) will become useful in the next section, where we will refer to $\mathsf{h}_\triangle$ as the *corresponding* A-*good function to* $\langle \Delta, T\rangle$.

The following lemma will be needed in our main proof in the next section.

Lemma 4.25 *If T is a subset of $\mathfrak{F}^+$ s.t. for every finite $T_0 \subseteq T$, $\prod T_0 \not\leq \mathsf{A}$, then $\langle \emptyset, T\rangle$ has the* A-*f.i.p.*

Proof. To prove that $\langle \emptyset, T\rangle$ has the A-f.i.p., it suffices to show that for every finite $T_0 \subseteq T$, there exists a good h s.t. $\prod\limits_{\mathsf{f} \in T_0} \mathsf{h}(\mathsf{f}) \not\leq \mathsf{h}(\mathsf{A})$. Since $\prod T_0 \not\leq \mathsf{A}$, or $\prod T_0 \Rightarrow \mathsf{A} \neq 1_{\mathfrak{F}^+}$

$(\mathcal{H}_2)$, there exists an $\mathfrak{s} \in \mathfrak{S}$ s.t. $\big(\prod T_0 \Rightarrow \mathsf{A}\big)(\mathfrak{s}) \neq \top$. Take $\mathsf{h} : \mathfrak{F}^+ \to \mathcal{H}$ be the function which for every $\mathsf{f} \in \mathfrak{F}^+$,

$$\mathsf{h}(\mathsf{f}) = \mathsf{f}(\mathfrak{s}).$$

In later terms (Def. 4.27), this function will be the *principal good function associated to* $\mathfrak{s}$ and it will be shown to be good (definition 4.27 and proposition 4.28). It is then direct that h is the required function, since $\mathsf{h}(\prod T_0 \Rightarrow \mathsf{A}) = \big(\prod T_0 \Rightarrow \mathsf{A}\big)(\mathfrak{s}) \neq \top$, thus $\prod_{\mathsf{f} \in T_0} \mathsf{h}(\mathsf{f}) \not\leq \mathsf{h}(\mathsf{A})$. ∎

After all these prerequisite definitions we are now ready to describe the canonical frame and model extensions. The apparent candidate for the set of states is the set of all good functions. We further need an accessibility relation (or function) which is next defined, and it can directly be checked that it is a proper generalization of the relation $\mathfrak{R}^{\mathfrak{ue}}$ in the classical case:

$$\mathfrak{g}^{\mathfrak{c}}(\mathsf{h}_1, \mathsf{h}_2) := \prod_{\mathsf{f} \in \mathfrak{F}^+} \Big(\mathsf{h}_2(\mathsf{f}) \Rightarrow \mathsf{h}_1(\mathrm{m}_{\mathfrak{g}}(\mathsf{f}))\Big) \cdot \prod_{\mathsf{f} \in \mathfrak{F}^+} \Big(\mathsf{h}_1(\mathrm{l}_{\mathfrak{g}}(\mathsf{f})) \Rightarrow \mathsf{h}_2(\mathsf{f})\Big)$$

Finally, the valuation function is given below, again extending the definition in the two-valued case:

$$v^{\mathfrak{c}}(\mathsf{h}, p) := \mathsf{h}([p]_{\mathfrak{M}}),$$

which satisfies: for every $a \in \mathcal{H}, v^{\mathfrak{c}}(\mathsf{h}, a) = a = \mathsf{h}([a])$, by Definitions 3.1 and 4.17.

Definition 4.26 Let $\mathfrak{F} = \langle \mathfrak{S}, \mathfrak{g} \rangle$ be a $\mathcal{H}$-modal frame and $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ the $\mathcal{H}$-modal model built on it. The *canonical frame and model extensions* associated to them are defined respectively as follows:

- $\mathfrak{F}^{\mathfrak{c}} = \langle \mathfrak{S}^{\mathfrak{c}}, \mathfrak{g}^{\mathfrak{c}} \rangle$
- $\mathfrak{M}^{\mathfrak{c}} = \langle \mathfrak{S}^{\mathfrak{c}}, \mathfrak{g}^{\mathfrak{c}}, v^{\mathfrak{c}} \rangle$,

where $\mathfrak{S}^{\mathfrak{c}}$ is the set of all good functions.

Of course, $v^{\mathfrak{c}}$ extends to all formulas of $L^{\mathcal{H}}_{\Box\Diamond}$ according to definition 3.2.

As a remark, it would be along with the stream of algebraic presentation to denote $\mathfrak{F}^{\mathfrak{c}}$ by $(\mathfrak{F}^+)_+$, and then prove duality theorems extending some of Ch.5 of [2]. However, that would be distracting for our current purpose, so we prefer to make do with just this simple argument.

We end up this section with the definition of another very important notion for the results in the next section, the analogous of a principal ultrafilter associated with a state $\mathfrak{s} \in \mathfrak{S}$. We leave again to the reader to check how the following definition properly extends that notion.

Definition 4.27 For every $\mathfrak{s} \in \mathfrak{S}$, the *principal good function associated to* $\mathfrak{s}$ is the function $\pi_{\mathfrak{s}} : \mathfrak{F}^+ \to \mathcal{H}$, s.t. for every $\mathsf{f} \in \mathfrak{F}^+$,

$$\pi_{\mathfrak{s}}(\mathsf{f}) = \mathsf{f}(\mathfrak{s})$$

The following proposition provides that for every $\mathfrak{s} \in \mathfrak{S}$, $\pi_{\mathfrak{s}}$ is an element of $\mathfrak{S}^{\mathfrak{c}}$.

Proposition 4.28 *For every $\mathfrak{s} \in \mathfrak{S}$, $\pi_{\mathfrak{s}}$ is a good function.*

PROOF. The proof is easy, because of the coordinate-wise way we defined the operators of $\mathfrak{F}^+$. If $\triangle$ and $\triangle_{\mathfrak{F}^+}$ are as in the proof of theorem 4.22, then for every f and g in $\mathfrak{F}^+$, $\pi_{\mathfrak{s}}(\mathsf{f}\triangle_{\mathfrak{F}^+}\mathsf{g}) = (\mathsf{f}\triangle_{\mathfrak{F}^+}\mathsf{g})(\mathfrak{s}) = \mathsf{f}(\mathfrak{s})\triangle\mathsf{g}(\mathfrak{s}) = \pi_{\mathfrak{s}}(\mathsf{f})\triangle\pi_{\mathfrak{s}}(\mathsf{g})$.

Also, for every $a \in \mathcal{H}$, $\pi_{\mathfrak{s}}([a]_{\mathfrak{M}}) = [a]_{\mathfrak{M}}(\mathfrak{s}) = a$. ∎

4.4.2 The main theorem

Two results are proved in this section. The first is described by the following sentence:

- *By identifying a state $\mathfrak{s}$ of a frame $\mathfrak{F}$ with its associated principal good function $\pi_{\mathfrak{s}}$, $\mathfrak{F}$ can (isomorphically) be seen as a (not necessarily generated) submodel of its canonical frame extension $\mathfrak{F}^{\mathfrak{c}}$.*

To prove this, we show that the accessibility function is preserved under the mapping : $\mathfrak{s} \mapsto \pi_{\mathfrak{s}}$.

Proposition 4.29 *For every $\mathfrak{s}, \mathfrak{t} \in \mathfrak{S}$, $\mathfrak{g}(\mathfrak{s}, \mathfrak{t}) = \mathfrak{g}^{\mathfrak{c}}(\pi_{\mathfrak{s}}, \pi_{\mathfrak{t}})$*

PROOF.

$$\begin{aligned}
\mathfrak{g}^{\mathfrak{c}}(\pi_{\mathfrak{s}}, \pi_{\mathfrak{t}}) &= \prod_{\mathsf{f} \in \mathfrak{F}^+} \Big(\pi_{\mathfrak{t}}(\mathsf{f}) \Rightarrow \pi_{\mathfrak{s}}(\mathrm{m}_{\mathfrak{g}}(\mathsf{f}))\Big) \cdot \prod_{\mathsf{f} \in \mathfrak{F}^+} \Big(\pi_{\mathfrak{s}}(\mathrm{l}_{\mathfrak{g}}(\mathsf{f})) \Rightarrow \pi_{\mathfrak{t}}(\mathsf{f})\Big) \\
&= \underbrace{\prod_{\mathsf{f} \in \mathfrak{F}^+} \Big(\mathsf{f}(\mathfrak{t}) \Rightarrow \mathrm{m}_{\mathfrak{g}}(\mathsf{f})(\mathfrak{s})\Big)}_{A} \cdot \underbrace{\prod_{\mathsf{f} \in \mathfrak{F}^+} \Big(\mathrm{l}_{\mathfrak{g}}(\mathsf{f})(\mathfrak{s}) \Rightarrow \mathsf{f}(\mathfrak{t})\Big)}_{B} \qquad \text{(Def. 4.27)}
\end{aligned}$$

We show

$$A \leq \mathfrak{g}(\mathfrak{s}, \mathfrak{t})$$

and hence,

$$\mathfrak{g}^{\mathfrak{c}}(\pi_{\mathfrak{s}}, \pi_{\mathfrak{t}}) \leq \mathfrak{g}(\mathfrak{s}, \mathfrak{t}).$$

Take $\mathsf{f} \in \mathfrak{F}^+$ s.t. $\mathsf{f}(\mathfrak{t}') = 1$ if $\mathfrak{t} = \mathfrak{t}'$, else $\mathsf{f}(\mathfrak{t}') = 0$. Then,

$$\begin{aligned}
A &= \prod_{\mathsf{f} \in \mathfrak{F}^+} \Big(\mathsf{f}(\mathfrak{t}) \Rightarrow \mathrm{m}_{\mathfrak{g}}(\mathsf{f})(\mathfrak{s})\Big) \\
&\leq \mathsf{f}(\mathfrak{t}) \Rightarrow \mathrm{m}_{\mathfrak{g}}(\mathsf{f})(\mathfrak{s}) \\
&\leq \mathrm{m}_{\mathfrak{g}}(\mathsf{f})(\mathfrak{s}) \\
&= \sum_{\mathfrak{t}' \in \mathfrak{S}} \mathfrak{g}(\mathfrak{s}, \mathfrak{t}') \cdot \mathsf{f}(\mathfrak{t}') \\
&= \mathfrak{g}(\mathfrak{s}, \mathfrak{t})
\end{aligned}$$

For the inverse, we show

$$\mathfrak{g}(\mathfrak{s}, \mathsf{t}) \le A$$

and

$$\mathfrak{g}(\mathfrak{s}, \mathsf{t}) \le B$$

For the first, it suffices to show $\mathfrak{g}(\mathfrak{s}, \mathsf{t}) \le \mathsf{f}(\mathsf{t}) \Rightarrow \mathrm{m}_{\mathfrak{g}}(\mathsf{f})(\mathfrak{s})$, for every $\mathsf{f} \in \mathfrak{F}^+$. We have:

$$\mathfrak{g}(\mathfrak{s}, \mathsf{t}) \cdot \mathsf{f}(\mathsf{t}) \le \mathrm{m}_{\mathfrak{g}}(\mathsf{f})(\mathfrak{s}) = \sum_{\mathsf{t}' \in \mathfrak{S}} \mathfrak{g}(\mathfrak{s}, \mathsf{t}') \cdot \mathsf{f}(\mathsf{t}'),$$

hence, $\mathfrak{g}(\mathfrak{s}, \mathsf{t}) \le \mathsf{f}(\mathsf{t}) \Rightarrow \mathrm{m}_{\mathfrak{g}}(\mathsf{f})(\mathfrak{s})$.

For the second, it suffices to show $\mathfrak{g}(\mathfrak{s}, \mathsf{t}) \le \mathrm{l}_{\mathfrak{g}}(\mathsf{f})(\mathfrak{s}) \Rightarrow \mathsf{f}(\mathfrak{s})$, for every $\mathsf{f} \in \mathfrak{F}^+$. We have:

$$\mathfrak{g}(\mathfrak{s}, \mathsf{t}) \cdot \prod_{\mathsf{t}' \in \mathfrak{S}} \Big(\mathfrak{g}(\mathfrak{s}, \mathsf{t}) \Rightarrow \mathsf{f}(\mathsf{t}')\Big) \le \mathfrak{g}(\mathfrak{s}, \mathsf{t}) \cdot \Big(\mathfrak{g}(\mathfrak{s}, \mathsf{t}) \Rightarrow \mathsf{f}(\mathsf{t})\Big) \le \mathsf{f}(\mathsf{t}),$$

hence,

$$\mathfrak{g}(\mathfrak{s}, \mathsf{t}) \le \Big(\prod_{\mathsf{t}' \in \mathfrak{S}} \mathfrak{g}(\mathfrak{s}, \mathsf{t}) \Rightarrow \mathsf{f}(\mathsf{t}')\Big) \Rightarrow \mathsf{f}(\mathsf{t}) = \mathrm{l}_{\mathfrak{g}}(\mathsf{f})(\mathfrak{s}) \Rightarrow \mathsf{f}(\mathsf{t})$$

∎

We straightforward continue with our main theorem, which establishes an ($\top$-)invariance result for the (stated) embedding of a model to its canonical model extension.

Theorem 4.30 *For every formula $\varphi \in L^{\mathcal{H}}_{\Box\Diamond}$ and good function $\mathsf{h} \in \mathfrak{S}^{\mathfrak{c}}$,*

$$v^{\mathfrak{c}}(\mathsf{h}, \varphi) = \mathsf{h}([\varphi]_{\mathfrak{M}}).$$

Hence, for every $\mathfrak{s} \in \mathfrak{S}$ and $\varphi \in L^{\mathcal{H}}_{\Box\Diamond}$,

$$v(\mathfrak{s}, \varphi) = v^{\mathfrak{c}}(\pi_{\mathfrak{s}}, \varphi).$$

That is, modal satisfaction is ($\top$-)*invariant under canonical extensions.*

Proof. Suppose that we have proved the first claim. Then the second one is derived by:

$$v^{\mathfrak{c}}(\pi_{\mathfrak{s}}, \varphi) = \pi_{\mathfrak{s}}([\varphi]_{\mathfrak{M}}) \overset{\text{Def. 4.27}}{=} [\varphi]_{\mathfrak{M}}(\mathfrak{s}) \overset{\text{Def. 4.14}}{=} v(\mathfrak{s}, \varphi).$$

In what follows the notion of principal good functions is no more being used. We will prove the first claim by induction on φ (for all good functions simultaneously).
For $\varphi = p \in \Phi$, by the definition of $v^{\mathfrak{c}}$, and for $\varphi = a \in \mathcal{H}$, by definitions 3.1 and 4.17.
For $\varphi = \varphi_1 \wedge \varphi_2$ or $\varphi_1 \vee \varphi_2$ or $\varphi_1 \supset \varphi_2$, we simply make use of (the definitions and) the fact that h is homomorphism. For instance:

$$
\begin{aligned}
v^{\mathfrak{e}}(\mathsf{h}, \varphi_1 \wedge \varphi_2) &= v^{\mathfrak{e}}(\mathsf{h}, \varphi_1) \cdot v^{\mathfrak{e}}(\mathsf{h}, \varphi_2) && \text{(by Def. 3.2)} \\
&= \mathsf{h}\big([\varphi_1]\big) \cdot \mathsf{h}\big([\varphi_2]\big) && \text{(by Ind. Hyp.)} \\
&= \mathsf{h}\big([\varphi_1] \cdot_{\mathfrak{F}^+} [\varphi_2]\big) && \text{(}\mathsf{h}\text{ is good)} \\
&= \mathsf{h}\big([\varphi_1 \wedge \varphi_2]\big)
\end{aligned}
$$

For each of the modal operators, we split our proofs into two parts. For $\Box\varphi$:

Part 1. $\mathsf{h}([\Box\varphi]_{\mathfrak{M}}) \leq v'(\mathsf{h}, \Box\varphi)$

$$
\begin{aligned}
v^{\mathfrak{e}}(\mathsf{h}, \Box\varphi) &= \prod_{\mathsf{h}' \in \mathfrak{S}^{\mathfrak{e}}} \Big(\mathfrak{g}^{\mathfrak{e}}(\mathsf{h}, \mathsf{h}') \Rightarrow v^{\mathfrak{e}}(\mathsf{h}', \varphi)\Big) \\
&= \prod_{\mathsf{h}' \in \mathfrak{S}^{\mathfrak{e}}} \Big(\mathfrak{g}^{\mathfrak{e}}(\mathsf{h}, \mathsf{h}') \Rightarrow \mathsf{h}'([\varphi]_{\mathfrak{M}})\Big) && \text{(by Ind. Hyp.)} \\
&= \prod_{\mathsf{h}' \in \mathfrak{S}^{\mathfrak{e}}} \left(\prod_{\mathsf{f} \in \mathfrak{F}^+} \Big(\mathsf{h}'(\mathsf{f}) \Rightarrow \mathsf{h}(\mathrm{m}_{\mathfrak{g}}(\mathsf{f}))\Big) \cdot \prod_{\mathsf{f} \in \mathfrak{F}^+} \Big(\mathsf{h}(\mathrm{l}_{\mathfrak{g}}(\mathsf{f})) \Rightarrow \mathsf{h}'(\mathsf{f})\Big) \Rightarrow \mathsf{h}'([\varphi]_{\mathfrak{M}})\right) \\
&\geq \prod_{\mathsf{h}' \in \mathfrak{S}^{\mathfrak{e}}} \left(\prod_{\mathsf{f} \in \mathfrak{F}^+} \Big(\mathsf{h}(\mathrm{l}_{\mathfrak{g}}(\mathsf{f})) \Rightarrow \mathsf{h}'(\mathsf{f})\Big) \Rightarrow \mathsf{h}'([\varphi]_{\mathfrak{M}})\right) && \text{(by } \mathcal{H}_4\text{)} \\
&\geq \prod_{\mathsf{h}' \in \mathfrak{S}^{\mathfrak{e}}} \left(\Big(\mathsf{h}(\mathrm{l}_{\mathfrak{g}}([\varphi]_{\mathfrak{M}})) \Rightarrow \mathsf{h}'([\varphi]_{\mathfrak{M}})\Big) \Rightarrow \mathsf{h}'([\varphi]_{\mathfrak{M}})\right) && \text{(fix } \mathsf{f}= [\varphi]_{\mathfrak{M}},\ \mathcal{H}_4\text{)} \\
&\geq \prod_{\mathsf{h}' \in \mathfrak{S}^{\mathfrak{e}}} \mathsf{h}(\mathrm{l}_{\mathfrak{g}}([\varphi]_{\mathfrak{M}})) && \text{(by lemma 2.3 (2))} \\
&= \mathsf{h}([\Box\varphi]_{\mathfrak{M}})
\end{aligned}
$$

Part 2. $v^{\mathfrak{e}}(\mathsf{h}, \Box\varphi) \leq \mathsf{h}([\Box\varphi]_{\mathfrak{M}})$

This is the most complicated part of the proof. Towards a contrary, suppose that for a fixed $\mathsf{h}_0 \in \mathfrak{S}^{\mathfrak{e}}$, $v^{\mathfrak{e}}(\mathsf{h}_0, \Box\varphi) \not\leq \mathsf{h}_0([\Box\varphi]_{\mathfrak{M}})$. Let T be the following subset of $\mathfrak{F}^+$:

$$T = \{\, [\mathsf{h}_0(\mathrm{l}_{\mathfrak{g}}(\mathsf{f}))] \Rightarrow \mathsf{f},\, \mathsf{z} \Rightarrow [\mathsf{h}_0(\mathrm{m}_{\mathfrak{g}}(\mathsf{z}))] : \mathsf{f}, \mathsf{z} \in \mathcal{H}^{\mathfrak{e}}\}.$$

Also let

$$\mathsf{A} = [v^{\mathfrak{e}}(\mathsf{h}_0, \Box\varphi)] \Rightarrow [\varphi]_{\mathfrak{M}}.$$

We first show that $\langle \emptyset, T\rangle$ has the A-f.i.p.: By Lemma 4.25, it suffices to show that for every finite $T_0 \subseteq T$, $\prod T_0 \not\leq \mathsf{A}$. If that was the case, there would exist finite subsets $\{\mathsf{f}_0, ..., \mathsf{f}_m\}$ and $\{\mathsf{z}_0, ..., \mathsf{z}_n\}$ of $\mathfrak{F}^+$, such that:

$$\bigwedge_{i=1,\ldots,m} \Big([\mathsf{h}_0(\mathrm{l}_{\mathfrak{g}}(\mathsf{f}_i))] \Rightarrow \mathsf{f}_i\Big) \cdot \bigwedge_{j=1,\ldots,n} \Big(\mathsf{z}_j \Rightarrow [\mathsf{h}_0(\mathrm{m}_{\mathfrak{g}}(\mathsf{z}_j))]\Big) \leq [v^{\mathfrak{e}}(\mathsf{h}_0, \Box\varphi)] \Rightarrow [\varphi]_{\mathfrak{M}}.$$

By lemma 4.15 (1, 2), we derive:

$$\bigwedge_{i=1,\ldots,m}\Big([\mathsf{h}_0(\mathrm{l}_{\mathfrak{g}}(\mathsf{f}_i))] \Rightarrow \mathrm{l}_{\mathfrak{g}}(\mathsf{f}_i)\Big) \cdot \bigwedge_{j=1,\ldots,n}\Big(\mathrm{m}_{\mathfrak{g}}(\mathsf{z}_j) \Rightarrow [\mathsf{h}_0(\mathrm{m}_{\mathfrak{g}}(\mathsf{z}_j))]\Big) \leq [v^{\mathfrak{C}}(\mathsf{h}_0, \Box\varphi)] \Rightarrow [\Box\varphi]_{\mathfrak{M}}.$$

Applying h_0 to each part of the inequality, we obtain:

$$\bigwedge_{i=1,\ldots,m}\Big(\mathsf{h}_0(\mathrm{l}_{\mathfrak{g}}(\mathsf{f}_i)) \Rightarrow \mathsf{h}_0(\mathrm{l}_{\mathfrak{g}}(\mathsf{f}_i))\Big) \cdot \bigwedge_{j=1,\ldots,n}\Big(\mathsf{h}_0(\mathrm{m}_{\mathfrak{g}}(\mathsf{z}_j)) \Rightarrow \mathsf{h}_0(\mathrm{m}_{\mathfrak{g}}(\mathsf{z}_j))\Big) \leq v^{\mathfrak{C}}(\mathsf{h}, \Box\varphi) \Rightarrow \mathsf{h}_0([\Box\varphi]_{\mathfrak{M}}),$$

since h_0 is a homomorphism and, further on, it maps every constant function $[a]$ to a. Hence,

$$\top \leq v^{\mathfrak{C}}(\mathsf{h}_0, \Box\varphi) \Rightarrow \mathsf{h}_0([\Box\varphi]_{\mathfrak{M}})$$

or

$$v^{\mathfrak{C}}(\mathsf{h}_0, \Box\varphi) \leq \mathsf{h}_0([\Box\varphi]_{\mathfrak{M}}),$$

which contradicts to our basic assumption.

Now, extend $\langle\emptyset, T\rangle$ to a maximal $\langle\Delta, T\rangle$ with the A-f.i.p., according to lemma 4.20, and take h_Δ be the A-good function corresponding to $\langle\Delta, T\rangle$. We will combine some 'facts' to reach a contradiction:

$$\begin{aligned}
v^{\mathfrak{C}}(\mathsf{h}_0, \Box\varphi) &= \prod_{\mathsf{h}\,\in\,\mathfrak{S}^{\mathfrak{C}}}\Big(\mathfrak{g}^{\mathfrak{C}}(\mathsf{h}_0, \mathsf{h}) \Rightarrow v^{\mathfrak{C}}(\mathsf{h}, \varphi)\Big) \\
&\leq \mathfrak{g}^{\mathfrak{C}}(\mathsf{h}_0, \mathsf{h}_\Delta) \Rightarrow v^{\mathfrak{C}}(\mathsf{h}_\Delta, \varphi) \\
&= \mathfrak{g}^{\mathfrak{C}}(\mathsf{h}_0, \mathsf{h}_\Delta) \Rightarrow \mathsf{h}_\Delta([\varphi]_{\mathfrak{M}}) \qquad \text{(Ind. Hyp.)}
\end{aligned}$$

and hence,

$$\begin{aligned}
\mathfrak{g}^{\mathfrak{C}}(\mathsf{h}_0, \mathsf{h}_\Delta) &\leq v^{\mathfrak{C}}(\mathsf{h}_0, \Box\varphi) \Rightarrow \mathsf{h}_\Delta([\varphi]_{\mathfrak{M}}) \\
&= \mathsf{h}_\Delta\Big([v^{\mathfrak{C}}(\mathsf{h}_0, \Box\varphi)] \Rightarrow [\varphi]_{\mathfrak{M}}\Big) \\
&= \mathsf{h}_\Delta(\mathsf{A}). \qquad (3)
\end{aligned}$$

Also,

$$\begin{aligned}
\mathfrak{g}^{\mathfrak{C}}(\mathsf{h}_0, \mathsf{h}_\Delta) &= \prod_{\mathsf{f}\,\in\,\mathfrak{F}^+}\Big(\mathsf{h}_\Delta(\mathsf{f}) \Rightarrow \mathsf{h}_0(\mathrm{m}_{\mathfrak{g}}(\mathsf{f}))\Big) \cdot \prod_{\mathsf{f}\,\in\,\mathfrak{F}^+}\Big(\mathsf{h}_0(\mathrm{l}_{\mathfrak{g}}(\mathsf{f})) \Rightarrow \mathsf{h}_\Delta(\mathsf{f})\Big) \\
&= \prod_{\mathsf{f}\,\in\,\mathfrak{F}^+}\mathsf{h}_\Delta\Big(\mathsf{f} \Rightarrow [\mathsf{h}_0(\mathrm{m}_{\mathfrak{g}}(\mathsf{f}))]\Big) \cdot \prod_{\mathsf{f}\,\in\,\mathfrak{F}^+}\mathsf{h}_\Delta\Big([\mathsf{h}_0(\mathrm{l}_{\mathfrak{g}}(\mathsf{f}))] \Rightarrow \mathsf{f}\Big) \\
&= \prod_{\mathsf{f}\,\in\,T}\mathsf{h}_\Delta(\mathsf{f}) \qquad (4)
\end{aligned}$$

(3) and (4) imply:

$$\prod_{\mathsf{f}\,\in\,T}\mathsf{h}_\Delta(\mathsf{f}) \quad \leq \quad \mathsf{h}_\Delta(\mathsf{A}).$$

Consider the finiteness of $\mathcal{H}$, hence for a finite $T_0 \subseteq T$ derive $\prod_{\mathsf{f} \in T} \mathsf{h}_\Delta(\mathsf{f}) = \prod_{\mathsf{f} \in T_0} \mathsf{h}_\Delta(\mathsf{f})$, and $\prod_{\mathsf{f} \in T_0} \mathsf{h}_\Delta(\mathsf{f}) \leq \mathsf{h}_\Delta(\mathsf{A})$, which contradicts to the fact that h is A-good for T (theorem 4.22).

Once we have proved the theorem for the $\Box$ operator, the $\Diamond$ case can easily be treated in a parallel way:

Part 1. $v^{\mathfrak{c}}(\mathsf{h}, \Diamond\varphi) \leq \mathsf{h}([\Diamond\varphi]_{\mathfrak{M}})$

$$\begin{array}{rcll}
v^{\mathfrak{c}}(\mathsf{h}, \Diamond\varphi) & = & \sum_{\mathsf{h}' \in \mathfrak{S}^{\mathfrak{c}}} \Big(\mathfrak{g}^{\mathfrak{c}}(\mathsf{h}, \mathsf{h}') \cdot v^{\mathfrak{c}}(\mathsf{h}', \varphi)\Big) & \\
& = & \sum_{\mathsf{h}' \in \mathfrak{S}^{\mathfrak{c}}} \Big(\mathfrak{g}^{\mathfrak{c}}(\mathsf{h}, \mathsf{h}') \cdot \mathsf{h}'([\varphi]_{\mathfrak{M}})\Big) & \text{(by Ind. Hyp.)} \\
& = & \sum_{\mathsf{h}' \in \mathfrak{S}^{\mathfrak{c}}} \Big(\prod_{\mathsf{f} \in \mathfrak{F}^+} \Big(\mathsf{h}'(\mathsf{f}) \Rightarrow \mathsf{h}(\mathrm{m}_{\mathfrak{g}}(\mathsf{f}))\Big) \cdot \prod_{\mathsf{f} \in \mathfrak{F}^+} \Big(\mathsf{h}(\mathrm{l}_{\mathfrak{g}}(\mathsf{f})) \Rightarrow \mathsf{h}'(\mathsf{f})\Big) \cdot \mathsf{h}'([\varphi]_{\mathfrak{M}})\Big) & \\
& \leq & \sum_{\mathsf{h}' \in \mathfrak{S}^{\mathfrak{c}}} \Big(\prod_{\mathsf{f} \in \mathfrak{F}^+} \Big(\mathsf{h}'(\mathsf{f}) \Rightarrow \mathsf{h}(\mathrm{m}_{\mathfrak{g}}(\mathsf{f}))\Big) \cdot \mathsf{h}'([\varphi]_{\mathfrak{M}})\Big) & \\
& \leq & \sum_{\mathsf{h}' \in \mathfrak{S}^{\mathfrak{c}}} \Big([\mathsf{h}'([\varphi]_{\mathfrak{M}}) \Rightarrow \mathsf{h}(\mathrm{m}_{\mathfrak{g}}([\varphi]_{\mathfrak{M}}))] \cdot \mathsf{h}'([\varphi]_{\mathfrak{M}})\Big) & (\text{fix } \mathsf{f}= [\varphi]_{\mathfrak{M}}) \\
& \leq & \sum_{\mathsf{h}' \in \mathfrak{S}^{\mathfrak{c}}} \mathsf{h}(\mathrm{m}_{\mathfrak{g}}([\varphi]_{\mathfrak{M}})) & \\
& = & \mathsf{h}([\Diamond\varphi]_{\mathfrak{M}}) &
\end{array}$$

Part 2. $\mathsf{h}([\Diamond\varphi]_{\mathfrak{M}}) \leq v^{\mathfrak{c}}(\mathsf{h}, \Diamond\varphi)$
Towards a contrary, suppose that for a fixed $\mathsf{h}_0 \in \mathfrak{S}^{\mathfrak{c}}$, $\mathsf{h}_0([\Diamond\varphi]_{\mathfrak{M}}) \not\leq v^{\mathfrak{c}}(\mathsf{h}_0, \Diamond\varphi)$. Let T be again the set:

$$T = \{\, [\mathsf{h}_0(\mathrm{l}_{\mathfrak{g}}(\mathsf{f}))] \Rightarrow \mathsf{f},\, \mathsf{z} \Rightarrow [\mathsf{h}_0(\mathrm{m}_{\mathfrak{g}}(\mathsf{z}))] : \mathsf{f}, \mathsf{z} \in \mathcal{H}^{\mathfrak{c}}\}$$

but

$$\mathsf{A} = [\varphi]_{\mathfrak{M}} \Rightarrow [v^{\mathfrak{c}}(\mathsf{h}_0, \Diamond\varphi)].$$

Following exactly the same steps as in the $\Box$ case, we can prove that $\langle \emptyset, T\rangle$ has the A-f.i.p.; if we suppose otherwise, we can derive the contradictory

$$\top \leq \mathsf{h}_0([\Diamond\varphi]_{\mathfrak{M}}) \Rightarrow v^{\mathfrak{c}}(\mathsf{h}_0, \Diamond\varphi).$$

Extend again $\langle \emptyset, T\rangle$ to a maximal $\langle \Delta, T\rangle$ with the A-f.i.p. and take the A-good function $\mathsf{h}_\Delta$corresponding to $\langle \Delta, T\rangle$. In analogy with the relation (3) of the $\Box$ case, here we have:

$$\begin{array}{rcll} v^{\mathfrak{c}}(\mathsf{h}_0, \Diamond\varphi) & = & \sum\limits_{\mathsf{h} \in \mathfrak{S}^{\mathfrak{c}}} \mathfrak{g}^{\mathfrak{c}}(\mathsf{h}_0, \mathsf{h}) \cdot v^{\mathfrak{c}}(\mathsf{h}, \varphi) & \\ & \geq & \mathfrak{g}^{\mathfrak{c}}(\mathsf{h}_0, \mathsf{h}_\Delta) \cdot v^{\mathfrak{c}}(\mathsf{h}_\Delta, \varphi) & \\ & = & \mathfrak{g}^{\mathfrak{c}}(\mathsf{h}_0, \mathsf{h}_\Delta) \cdot \mathsf{h}_\Delta([\varphi]_{\mathfrak{M}}) & \text{(Ind. Hyp.)} \end{array}$$

and hence,

$$\begin{array}{rcll} \mathfrak{g}^{\mathfrak{c}}(\mathsf{h}_0, \mathsf{h}_\Delta) & \leq & \mathsf{h}_\Delta([\varphi]_{\mathfrak{M}}) \Rightarrow v^{\mathfrak{c}}(\mathsf{h}_0, \Diamond\varphi) & \\ & = & \mathsf{h}_\Delta\Big([\varphi]_{\mathfrak{M}}) \Rightarrow [v^{\mathfrak{c}}(\mathsf{h}_0, \Diamond\varphi)]\Big) & \\ & = & \mathsf{h}_\Delta(\mathsf{A}). & (5) \end{array}$$

(5) and (4) give again:

$$\prod_{\mathsf{f} \in T} \mathsf{h}_\Delta(\mathsf{f}) \quad \leq \quad \mathsf{h}_\Delta(\mathsf{A}),$$

which, by the same reasoning, contradicts to the fact that h is A-good for T. ∎

We close the section with the *anti*-invariance result on the frame level.

Theorem 4.31 *Let $\mathfrak{F} = \langle \mathfrak{S}, \mathfrak{g} \rangle$ be an $\mathcal{H}$-frame, $\mathfrak{F}^{\mathfrak{c}} = \langle \mathfrak{S}^{\mathfrak{c}}, \mathfrak{g}^{\mathfrak{c}} \rangle$ its canonical extension and X a formula of $L^{\mathcal{H}}_{\Box\Diamond}$. Then*

$$\mathfrak{F}^{\mathfrak{c}} \Vdash X \textit{ implies } \mathfrak{F} \Vdash X.$$

Proof. By contraposition, assume $\mathfrak{F} \not\Vdash X$. That is, there is a valuation v and a state $\mathfrak{s} \in \mathfrak{S}$ s.t. $v(\mathfrak{s}, X) \neq \top$. Then, by theorem 4.30, $v^{\mathfrak{c}}(\pi_{\mathfrak{s}}, X) \neq \top$. Hence, $\mathfrak{F}^{\mathfrak{c}} \not\Vdash X$. ∎

4.5 A brief digression: Algebras & General Frames for mv-ML

In this mini subsection we discuss briefly the algebraic semantics of $L^{\mathcal{H}}_{\Box\Diamond}$, barely touched in our investigations. We have already done this at many points throughout the text, but it seems to be worthwhile to gather all these results together and comment on them uniformly.

Given an $\mathcal{H}$-frame $\mathfrak{F} = \langle \mathfrak{S}, \mathfrak{g} \rangle$, the structure

$$\mathfrak{F}^{+} = \big\langle \langle \mathcal{H}^{\mathfrak{S}}, +_{\mathcal{H}^{\mathfrak{S}}}, \cdot_{\mathcal{H}^{\mathfrak{S}}}, \Rightarrow_{\mathcal{H}^{\mathfrak{S}}}, 0_{\mathcal{H}^{\mathfrak{S}}} \rangle, \mathrm{l}_{\mathfrak{g}}, \mathrm{m}_{\mathfrak{g}} \big\rangle$$

is a *Heyting algebra with operators*, the exact bimodal equivalent of a full complex algebra. The operators $\mathrm{l}_{\mathfrak{g}}$ and $\mathrm{m}_{\mathfrak{g}}$ serve for the interpretation of the modal operators of the modal operators $\Box$ and $\Diamond$; lemma 4.15 (3) & (4) asserts that the operators satisfy two basic axioms of the proof theory of mv-MLs (cf. [7]) and it is easy to verify that **K**. $\Box(A \supset B) \supset (\Box A \supset \Box B)$ is satisfied as well. Everything seems perfect, with a sole exception: we have to guarantee in this semantics that every $a \in \mathcal{H}$ is interpreted on $[a]_{\mathfrak{M}}$

and thus $[a]_{\mathfrak{M}}$ implicitly becomes a part of the similarity type, if our intended 'algebraic' semantics owe to be well-behaved. In that way, the encoding of $\mathcal{H}$ in $L^{\mathcal{H}}_{\Box\Diamond}$ which provides great expressive power (cf. the weak model axioms in [16, 14]), practically trivializes the most straightforward attempt to algebraize mv-MLs. At a first glance, we hoped that we would be able to provide an interesting algebraic semantics for the modal languages built on subdirectly irreducible HA s (cf. [1]) but the above observations indicate that such an attempt leads to more-or-less trivial results.

However, we have gained a small profit. En route, we have set the tools for defining a decent notion of a *general $\mathcal{H}$-frame*. Much obviously by now, such a structure consists of an $\mathcal{H}$-frame, accompanied by a set of admissible valuations: a subalgebra of the power algebra $\mathcal{H}^{\mathfrak{S}}$ (which means that it contains $0_{\mathcal{H}^{\mathfrak{S}}}, 1_{\mathcal{H}^{\mathfrak{S}}}$ and is closed under the modal operators $\mathrm{l}_{\mathfrak{g}}$, $\mathrm{m}_{\mathfrak{g}}$). An $\mathcal{H}$-model built on a general $\mathcal{H}$-frame will then require that for each $p \in \Phi$, $[p]_{\mathfrak{M}} = \mathsf{f}$, for some 'admissible' $\mathsf{f} \in \mathcal{H}^{\mathfrak{S}}$. It would be much interesting to see what results can be obtained by developing the theory of general $\mathcal{H}$-frames for mv-MLs.

5 Multiple-Expert Modal Frames and Constructions

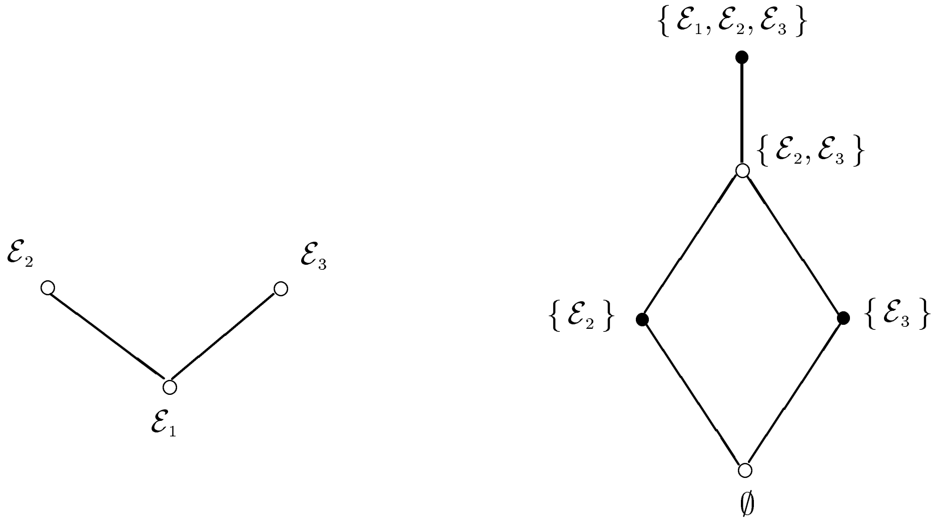
In this section we work in the class of many-valued modal languages built on *finite* HAs. Each language of this class can be reformulated in a way that is of interest to KR situations involving many interrelated experts. We shall provide in this section a brief description of this facet of Fitting's many-valued modal languages but the interested reader should refer to [7, Sect. 1, 3 & 5] for technical details. So, assume an epistemic situation involving (possibly infinitely) many states of affairs (possible worlds) and (finitely) many experts, whose opinion we value. Obviously, if the experts are independent there is nothing genuinely interesting to deal with. The scenario becomes interesting when the experts are related through a sort of hierarchical organization, which affects the way each of them makes up her mind on the truth of certain statements: some experts dominate others. Reasonably enough, we consider that the binary dominance relation is antisymmetric and transitive, and is broadly conceived: every experts dominates herself, that is the relation is also reflexive. In simple words, a partial order relation is imposed on the experts. Before proceeding to a technical description of the epistemic situations, an example is in order.

Assume we have three experts $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, related as in Fig. 1.(i): $\mathcal{E}_1$ dominates $\mathcal{E}_2$ and $\mathcal{E}_3$[3]. Further on, assume a set of possible worlds, common to all agents and a common epistemic language. We require that the dominance relation has the following effect on the accessibility relation each expert considers on the set of possible worlds:

D_1: an expert $\mathcal{E}_i$ considers that $\mathfrak{s}_1$ 'sees' $\mathfrak{s}_2$, iff every expert dominated by $\mathcal{E}_i$ does so

Note that each expert provides us a classical relational frame but, as a whole, the experts'

[3]It would be perhaps more intuitive to represent the hierarchically ordered experts, by placing the 'dominating' above the 'dominated' ones. We have chosen however to follow the standard mathematical practice and represent both the partial ordering of Fig. 1.(i), as well as the lattice of Fig. 1.(ii) with their Hasse diagrams; this will facilitate the discussion on the transitions between these two structures.



(i) (ii)

Figure 1: From many experts (i), to many truth values (ii), and vice versa

accessibility relations should satisfy condition D_1. We can proceed from multiple-expert frames to models, by providing a valuation. Again, we impose a similar requirement, which reasonably reflects the effect of dominance in truth assignments to propositional variables, for every expert, inside every possible world:

D_2: what an expert $\mathcal{E}_i$ accepts to be true in a possible world, must also be accepted by every expert $\mathcal{E}_i$ dominates

The rules for extending the valuation to all formulas of the epistemic language, also reflect the dominance relation. Formulas of the form $A \wedge B$ and $A \vee B$ are treated locally by an expert inside a possible world, in the obvious way. On the other hand, formulas of the form $\neg A$, $A \supset B$, $\Box A$, $\Diamond A$, are set to true by an expert inside a possible world, iff every dominated agent considers them to be true also in this possible world. A flavour of intuitionistic logic is more than obvious.

We are interested now in finding the experts' 'consensus', that is, in elegantly calculating the modal formulas on which our experts agree. This problem can be reformulated as one involving a many-valued language, where sets of experts who agree on the truth of an epistemic statement can be seen as generalized truth values. For instance, to denote that all experts agree on the truth of $\Box A$, we say that the truth value of $\Box A$ is $\{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$. Note however an important point: by D_2, **not every set of experts is an 'admissible' generalized truth value**. For instance, it cannot happen that $\mathcal{E}_1$ alone considers A to be true. The 'admissible' sets of experts are those which are dominance-closed, that is, upwards-closed in ordered sets terminology. Similarly for the accessibility relation between states: it can be given a generalized truth value (the set of experts who agree on the existence of an edge) and 'admits' (by D_1) the same sets of experts as such. To complete the example, it remains to notice that the 'admissible' sets of experts provide us a finite Heyting algebra, when ordered under set inclusion:

the lattice of sets of Fig. 1.(ii). A straightforward proof verifies that the rules (we informally sketched above) for extending the valuation to all formulas preserve requirement D_2, and thus this kind of multiple-expert modal models 'captures' the dominance relation. Actually, the multiple-expert situation described can be entirely translated to a many-valued modal model of the language $L^{\mathcal{H}}_{\Box\Diamond}$, where $\mathcal{H}$ is the lattice of Fig. 1.(ii); for a detailed proof of this claim see [7, Sect. 5]. The other translation is also feasible: as a first step, the reader can check that we recover the dominance relation of Fig. 1.(i) by ordering (under set inclusion) the *prime* filters of the lattice in Fig. 1.(ii)[4].

In general, multiple-expert epistemic situations of the kind sketched above are formally described by multiple-expert modal frames of the form $\langle \mathcal{E}, \mathcal{D}, \mathcal{G}, \mathcal{R} \rangle$, where: $\mathcal{E}$ is a finite non-empty set of experts, $\mathcal{D}$ is the partial-order dominance relation on $\mathcal{E}$, $\mathcal{G}$ is a non-empty set of possible worlds, and $\mathcal{R}$ is an $\mathcal{E}$-indexed set of accessibility relations on $\mathcal{G}$ satisfying requirement D_1 [7, Sect. 3]. We construct a multiple-expert modal model by providing a valuation v that respects D_2, and in a similar fashion, we extend it to all modal formulas (as sketched above). In [7] the equivalence of multiple-expert modal models with the $\mathcal{H}$-modal models of Def. 3.1 is established and thus we speak of an equivalent formulation of these logics. The starting point for proving this equivalence is actually G. Birkhoff's theorem on the dual representation between finite partial orders and finite distributive lattices (that is, finite HAs), as in our example above; see [5, 8.17-8.19] for a proof stated in terms of down-sets and prime ideals. In [7] this is significantly extended, actually lifting to the modal setting the proofs on the equivalence between the algebraic and the possible-world semantics of propositional intuitionistic logic. The technical construction is very elegant and at the same time a very interesting example of application of mathematical ideas in AI problems.

We are now in the position to express the meaning of our results in this setting. Note first, that this new interpretation is concerned with the 'a' assigned to the various notions of this paper. These are the a-constructions we have seen, the a-invariance results following and the several non-definability results derived and involving the value a. Of course all these occur in both the frame and model level.

Then, it suffices to observe that the finite HA of the 'admissible' subsets of experts is a bounded lattice of sets; thus, it is ordered by set inclusion, meet is set intersection and join is set union. It is now fairly easy to identify the meaning of our conditions in the multiple-expert scenario. To start with an example, let us explain the meaning of $\mathfrak{M}' = \langle \mathfrak{S}', \mathfrak{g}', v' \rangle$ being a-generated submodel of $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ (or $\mathfrak{F}' = \langle \mathfrak{S}', \mathfrak{g}' \rangle$ a-generated subframe of $\mathfrak{F} = \langle \mathfrak{S}, \mathfrak{g} \rangle$) . Given the details above, for a multiple-expert frame, the properties 1, 2 and $(a - cc)$ of definition 4.4 take the form:

1. for every $\mathfrak{s}, \mathfrak{t} \in \mathfrak{S}'$, $a \cap \mathfrak{g}'(\mathfrak{s}, \mathfrak{t}) = a \cap \mathfrak{g}(\mathfrak{s}, \mathfrak{t})$
2. for every $\mathfrak{s} \in \mathfrak{S}'$ and $p \in \Phi$, $a \cap v(\mathfrak{s}, p) = a \cap v'(\mathfrak{s}, p)$.
3. if $\mathfrak{s} \in \mathfrak{S}'$ and $a \cap \mathfrak{g}(\mathfrak{s}, \mathfrak{t}) \neq \bot$, then $\mathfrak{t} \in \mathfrak{S}'$ $\qquad (a - cc)$

[4]Note that in finite HAs, prime filters are obtained as the principal filters generated by the join-irreducible elements [5, Lem. 8.16, Prop. 9.4]. The join-irreducible elements of the lattice in Fig. 1.(ii) are shaded.

Keeping in mind that a is a set of experts, $\mathfrak{g}(\mathfrak{s},\mathfrak{t})$ denotes the set of experts who agree that $\mathfrak{s}$ 'sees' $\mathfrak{t}$, $v(\mathfrak{s},\varphi)$ is the set of experts who accept φ to be true in $\mathfrak{s}$ and that we coincide $\mathcal{H}$-frames and models with their corresponding multiple expert ones, it is easy to see that these conditions assert that:

1′. *for every* $\mathfrak{s},\mathfrak{t}\in\mathfrak{S}'$, *every expert in* a *considers that* $\mathfrak{s}$ *'sees'* $\mathfrak{t}$ *in frame* $\mathfrak{F}'$ *iff she considers that* $\mathfrak{s}$ *'sees'* $\mathfrak{t}$ *in frame* $\mathfrak{F}$.

2′. *for every* $\mathfrak{s},\mathfrak{t}\in\mathfrak{S}'$, *every expert in* a *accepts* φ *to be true in* $\mathfrak{s}$ *in model* $\mathfrak{M}'$ *iff she accepts* φ *to be true in* $\mathfrak{s}$ *in model* $\mathfrak{M}$.

3′. *if* $\mathfrak{s}\in\mathfrak{S}'$ *and there is at least one expert in* a *who considers that* $\mathfrak{s}$ *'sees'* $\mathfrak{t}$ *in frame* $\mathfrak{F}$, *then* $\mathfrak{t}\in\mathfrak{S}'$ $\qquad (a-cc)'$.

In other words, it is demanded for the conditions of generated submodels of the classical case to hold *for every expert in a*. The same interpretation can be given for the constructions of a-disjoint unions and a-bounded morphisms. For all these, the a-invariance result in the model level claims that the corresponding condition concerning the valuation extends to all formulas of $L^{\mathcal{H}}_{\Box\Diamond}$ and can be explained in a similar to the $2'$ way. For the a-invariance result on the frame level, the crucial point of e.g. theorem 4.10 could be (by a simple reasoning) translated:

- *For every* $X\in L^{\mathcal{H}}_{\Box\Diamond}$, *if all experts in* a *consider that* X *is valid in* $\mathfrak{F}$, *then all experts in* a *consider that* X *is valid in* $\mathfrak{F}'$.

Note that the same translation would be derived if we had considered the equivalent version of frame a-invariance, as stated in the beginning of section 4; it is hence better shown why those two versions ought to be equivalent.

Finally, let's try to interpret the (frame) non-definability result of example 4.11:

- *The class* C *of all multiple expert modal frames in which every expert in a considers that she 'sees' herself 'with value a' is not modally definable.*

As a remark, note that the proofs of our results throughout the paper can be carried out in the context of multiple-expert frames; this can be done along the lines of [7, Sect. 5]. However, this would not provide easier proofs in general and would limit the scope of our theorems in the class of languages built on finite HAs.

6 Conclusion

In this paper we have developed a significant part of the model theory of many-valued modal logics, focusing on the most important frame constructions. There exist several reasons for considering such a project, on the model theory of mv-MLs, worth pursuing.

- *an independent mathematical interest.* It is natural to develop mathematical structures in a broader context and this usually assists in understanding things better in the classical case. Moreover, it allows for testing the robustness of techniques and constructions which is an additional indication for their formal value. Many-valued modal logics provide a good context for developing a (version of a) unifying modal model theory with a strong algebraic flavor.

- *the potential for applications in AI* provided by the equivalent multiple-expert semantics. Multi-agent systems research is at a premium nowadays, as emphasis is paid on situations with complex intelligent entities collaborating in heterogenerous environments. Applications include distributed systems, formal verification, etc. In that respect, mv-MLs are useful items in the rich and varying collection of non-classical logics in AI.

- *an interesting perspective on model theory.* This family of logics embodies an interesting mix of lattice theory and logic, actually inherited from its semantics which combine Kripke intuitionistic and modal frames [7]. Thus, setting up the model theory of mv-MLs imports some sophisticated techniques from algebra, at least if one is willing to accept that Universal Algebra and Lattice theory provide some level of sophistication in logical investigations. Actually we believe that mv-MLs provide a context in which the classical Chang & Keisler equation "model theory = universal algebra + logic" finds a very natural incarnation.

There exist several ways in which this research can be extended. We mention here a few. Firstly, a bisimulation-based analysis of these logics is awaiting further research. This is also associated with the issue of finding a 'classical' corresponding language, at least for subclasses of HAs (for instance the finite chains) and provide the equivalents on the van-Benthem characterization theorem (for $\mathcal{H}$-models) and the Goldblatt-Thomson theorem (for $\mathcal{H}$-frames). Also, an algebraic generalization of ultraproducts of $\mathcal{H}$-frames (and models) seems feasible, with the aim of providing a Łos-like theorem.

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